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ON FINITE DIFFERENCE INEQUALITY OF LYAPUNOV TYPE

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1. Introduction

The classical inequality of Lyapunov [1] states that if $y(t)$ is a nontrivial solution of the differential system

$$y''(t)+p(t)y(t) = 0, \qquad a\le t\le b,$$
$$y(a) = y(b) = 0,$$

where $p(t)$ is a real and continuous function defined on [a,b], then

$$(b-a)\int_a^b |p(t)|dt > 4.$$

Write throughout $\Delta x(n) = x(n+1)-x(n)$.

Pachaptte has establihed in [2] the following discrete analogue of Lyapunov type inequality:

Theorem A. *Let $p(n)$ and $r(n)$ be real-valued functions defined on $I = \{a, a+1, a+2.\cdots, b\}$, where a, b are integers, and $r(n)>0$ for $n\in I$. If $x(n)$ is a solution of the equation*

$$\Delta(r(n)\Delta x(n))+p(n)x(n) = 0, \tag{1}$$

such that $x(a)=x(b)=0$, and $x(n)\neq 0$, $\forall n\in I^0=\{a+1,a+2,\cdots,b-1\}$, then

$$4\le\Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)\Big(\sum_{n=a}^{b-2}|p(n)|\Big). \tag{2}$$

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The main purpose of this paper is to given some generalizations of Theorem A.

2. Preliminary Results

In order to prove our main results, we need the follwoing lemmas:

Lemma 1. If u, v are any functions defined on I, and m is a positive integer then

$$\sum_{n=a}^{b-1} u^m(n)\Delta v(n) \;=\; u^m(b-1)v(b) - u^m(a)v(a) - \sum_{n=a}^{b-2} v(n+1)\Delta u(n)w(n)$$

where $w(n) = u^{m-1}(n+1) + u^{m-2}(n+1)u(n) + \cdots + u(n+1)u^{m-2}(n) + u^{m-1}(n)$.

Proof. We have

$$\begin{aligned}
&\sum_{n=a}^{b-1} u^m(n)\Delta v(n)\\
&= \sum_{n=a}^{b-1} u^m(n)(v(n+1) - v(n))\\
&= \sum_{n=a}^{b-1} u^m(n)v(n+1) - \sum_{n=a}^{b-1} u^m(n)v(n)\\
&= \sum_{n=a}^{b-1} u^m(n)v(n+1) - \sum_{n=a-1}^{b-2} u^m(n+1)v(n+1)\\
&= u^m(b-1)v(b) - u^m(a)v(a) - \sum_{n=a}^{b-2}\Big(u^m(n+1) - u^m(n)\Big)v(n+1).
\end{aligned}$$

Since

$$\begin{aligned}
&u^m(n+1) - u^m(n)\\
&= \Big(u(n+1) - u(n)\Big)\Big(u^{m-1}(n+1) + u^{m-2}(n+1)u(n) + \cdots\\
&\quad + u(n+1)u^{m-2}(n) + u^{m-1}(n)\Big)\\
&= \Delta u(n)w(n).
\end{aligned}$$

it follows that

$$\sum_{n=a}^{b-1} u^m(n)\Delta v(n) = u^m(b-1)v(b) - u^m(a)v(a) - \sum_{n=a}^{b-2} v(n+1)\Delta u(n)w(n).$$

Lemma 2. *Under the hypotheses of lemma 1, if* $u(n) = r(n)\Delta x(n)$, *and* $M = \max_{k\in 1} |x(k)|$, *then*

$$|w(n)| \le (2M)^{m-1} \sum_{i=1}^{m} r^{m-i}(n+1)r^{i-1}(n).$$

Proof. We have

$$\begin{aligned} w(n) &= u^{m-1}(n+1) + u^{m-2}(n+1)u(n) + \cdots + u(n+1)u^{m-2}(n) + u^{m-1}(n) \\ &= \sum_{i=1}^{m} u^{m-i}(n+1)u^{i-1}(n) \\ &= \sum_{i=1}^{m} (r(n+1)\Delta x(n+1))^{m-i}(r(n)\Delta x(n))^{i-1}, \end{aligned}$$

so that

$$\begin{aligned} |w(n)| &\le \sum_{i=1}^{m} r^{m-i}(n+1)|\Delta x(n+1)|^{m-i} r^{i-1}(n)|\Delta x(n)|^{i-1} \\ &\le (2M)^{m-1} \sum_{i=1}^{m} r^{m-i}(n+1)r^{i-1}(n). \end{aligned}$$

3. Main Results

Theorem 1. *Let* $x(n)$ *be a solution of the equation (1) such that* $x(a) = x(b) = 0$, *and* $x(n) \neq 0$, $\forall n \in I^0$. *If* m *is a nonnegative integer, then*

$$4 \le \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)^{2m+1} \Big(\sum_{n=a}^{b-2} |p(n)| \sum_{i=1}^{2m+1} r^{2m+1-i}(n+1)r^{i-1}(n)\Big). \qquad (3)$$

Proof. Let $M = \max_{k\in I^0} |x(k)|$. Then $M > 0$, since $x(n) \neq 0$, $\forall n \in I^0$. From the following identities:

$$x(k) = \sum_{n=a}^{k-1} \Delta x(n), \qquad k \in I^0.$$

$$x(k) = -\sum_{n=k}^{b-1} \Delta x(n), \qquad k \in I^0,$$

it follows that

$$2|x(k)| \leq \sum_{n=a}^{b-1} |\Delta x(n)|, \qquad k \in I^0.$$

Hence

$$2M \leq \sum_{n=a}^{b-1} |\Delta x(n)|,$$

so that

$$(2M)^{2m+2} \leq \Big(\sum_{n=a}^{b-1} r^{\frac{-2m-1}{2m+2}}(n) r^{\frac{2m+1}{2m+2}}(n)|\Delta x(n)|\Big)^{2m+2}.$$

Using Hölder's inequality with indices $\frac{2m+2}{2m+1}$, $2m+2$ we have

$$\begin{aligned}(2M)^{2m+2} &\leq \Big(\sum_{n=a}^{b-1}(r^{\frac{-2m-1}{2m+2}}(n))^{\frac{2m+2}{2m+1}}\Big)^{2m+1}\Big(\sum_{n=a}^{b-1}(r^{\frac{2m+1}{2m+2}}(n)|\Delta x(n)|)^{2m+2}\Big)\\ &= \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-1} r^{2m+1}(n)|\Delta x(n)|^{2m+2}\Big)\\ &= \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-1}(r(n)\Delta x(n))^{2m+1}\Delta x(n)\Big).\end{aligned}$$

It follows from Lemma 1 that

$$(2M)^{2m+2} \leq \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(-\sum_{n=a}^{b-2}(x(n+1)\Delta(r(n)\Delta x(n))w(n))\Big), \tag{4}$$

where $w(n) = \sum_{i=o}^{2m}(r(n+1)\Delta x(n+1))^{2m-i}(r(n)\Delta x(n))^i$, Since $\Delta(r(n)\Delta x(n)) =$

$-p(n)x(n)$, it follows from (4), and lemma 2 that

$$
\begin{aligned}
(2M)^{2m+2} &\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2}x(n+1)p(n)x(n)w(n)\Big)\\
&\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2}|x(n+1)||p(n)||x(n)||w(n)|\Big)\\
&\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}M^2\Big(\sum_{n=a}^{b-2}|p(n)||w(n)|\Big)\\
&\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}M^2\Big(\sum_{n=a}^{b-2}|p(n)|(2M)^{2m}\sum_{i=1}^{2m+1}r^{2m+1-i}(n)r^{i-1}(n)\Big).
\end{aligned}
$$

Therefore

$$4\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2}|p(n)|\sum_{i=1}^{2m+1}r^{2m+1-i}(n+1)r^{i-1}(n)\Big).$$

This completes the proof of the theorem.

Remark 1. In the special case $m=0$, the inequality (3) reduces to the inequality (2) in Theorem A.

Theorem 2. *Let $p_1(n)$, $p_2(n)$, $r(n)$ be real-valued functions defined on I, and $r(n)>0$ for $n\in I=\{a,a+1,\cdots,b\}$. Let $x(n)$ be a solution of the equation $\Delta(r(n)\Delta x(n))+p_1(n)\Delta x(n)+p_2(n)x(n)=0$ such that $x(a)=x(b)=0$ and $x(n)\neq 0$, $\forall n\in I^0$. If m is a nonnegative integer, then*

$$4\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2}(2|p_1(n)|+|p_2(n)|)\sum_{i=1}^{2m+1}r^{2m+1-i}(n+1)r^{i-1}(n)\Big).\quad (5)$$

Proof. As in the proof of theorem 1, we have

$$(2M)^{2m+2}\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(-\sum_{n=a}^{b-2}x(n+1)\Delta(r(n)\Delta x(n))w(n)\Big)$$

Since $-\Delta(r(n)\Delta x(n)) = p_1(n)\Delta x(n) + p_2(n)x(n)$, it follows that

$$\begin{aligned}
(2M)^{2m+2} \leq& \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2} x(n+1)(p_1(n)\Delta x(n) + p_2(n)x(n))w(n)\Big)\\
\leq& \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2} |x(n+1)|(|p_1(n)||\Delta x(n)|\\
&+ |p_2(n)||x(n)|)|w(n)|\Big)\\
\leq& \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2} M(|p_1(n)|2M + |p_2(n)|M)|w(n)|\Big)\\
\leq& \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2} M^2(2|p_1(n)| + |p_2(n)|)(2M)^{2m}\\
&\sum_{i=1}^{2m+1} r^{2m+1-i}(n+1)r^{i-1}(n)\Big).
\end{aligned}$$

Thus

$$4 \leq \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\sum_{n=a}^{b-2}(2|p_1(n)| + |p_2(n)|)\sum_{i=1}^{2m+1} r^{2m+1-i}(n+1)r^{i-1}(n)\Big).$$

Remark 2. The inequality (5) reduces to the inequality (3) when $p_1(n) \equiv 0$.

Theorem 3. *Let $x(n)$ be a solution of the equation (1) such that $x(a) = \sigma_1 x(a+1)$, $x(b) = \sigma_2 x(b-1)$, where $-1 \leq \sigma_1, \sigma_2 \leq 1$, and $x(n) \neq 0$. $\forall n \in I^0$. If m is a nonnegative integer, then*

$$\begin{aligned}
&(2 - |\sigma_1| - |\sigma_2|)^{2m+2} \qquad (6)\\
\leq& \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(r^{2m+1}(b-1)(1-\sigma_2)^{2m+1}|\sigma_2|\\
&+ r^{2m+1}(a)(1-\sigma_1)^{2m+1}|\sigma_1| + \sum_{n=a}^{b-2}|p(n)|2^{2m}\sum_{i=1}^{2m+1} r^{2m+1-i}(n+1)r^{i-1}(n)\Big)
\end{aligned}$$

Proof. Let $M = \max_{k\in I}|x(k)|$. Then $M > 0$, since $x(n) \neq 0$, $\forall n \in I^0$. We have

$$x(k) = \sum_{n=a}^{k-1}\Delta x(n) + x(a), \text{ and} x(k) = x(b) - \sum_{n=k}^{b-1}\Delta x(n),$$

so that

$$|x(k)| \leq \sum_{n=a}^{k-1} |\Delta x(n)| + |x(a)| \text{ and} |x(k)| \leq x(b) + \sum_{n=k}^{b-1} |\Delta x(n)|.$$

Hence

$$\begin{aligned} 2|x(k)| &\leq \sum_{n=a}^{b-1} |\Delta x(n)| + |x(a)| + |x(b)| \\ &= \sum_{n=a}^{b-1} |\Delta x(n)| + |\sigma_1||x(a+1)| + |\sigma_2||x(b-1)| \\ &\leq \sum_{n=a}^{b-1} |\Delta x(n)| + (|\sigma_1| + |\sigma_2|)M, \qquad k \in I, \end{aligned}$$

which gives

$$2M \leq \sum_{n=a}^{b-1} |\Delta x(n)| + (|\sigma_1| + |\sigma_2|)M.$$

On substraction and rise to the power $2m+2$, where $m > 0$, we have

$$\begin{aligned} (2 - |\sigma_1| - |\sigma_2|)^{2m+2} M^{2m+2} &\leq \Big(\sum_{n=a}^{b-1} |\Delta x(n)|\Big)^{2m+2} \\ &= \Big(\sum_{n=a}^{b-1} r^{\frac{-2m-1}{2m+2}}(n) r^{\frac{2m+1}{2m+2}}(n) |\Delta x(n)|\Big)^{2m+2} \end{aligned}$$

Using lemma 1, lemma 2 and applying Hölder's inequality to infer

$$\begin{aligned} &(2-|\sigma_1| - |\sigma_2|)^{2m+2} M^{2m+2} \\ &\leq \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)^{2m+1} \Big(\sum_{n=a}^{b-1} r^{2m+1}(n) |\Delta x(n)|^{2m+2}\Big) \\ &= \Big(\sum_{n=a}^{b-1} \frac{1}{r(n)}\Big)^{2m+1} \Big(\sum_{n=a}^{b-1} (r(n)\Delta x(n))^{2m+2} \Delta x(n)\Big) \end{aligned}$$

$$= \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\big(r(b-1)\Delta x(b-1)\big)^{2m+1}x(b) - \big(r(a)\Delta x(a)\big)^{2m+1}x(a)$$
$$- \sum_{n=a}^{b-2} x(n+1)\Delta\big(r(n)\Delta x(n)\big)w(n)\Big)$$
$$= \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(\big(r(b-1)\Delta x(b-1)\big)^{2m+1}x(b) - \big(r(a)\Delta x(a)\big)^{2m+1}x(a)$$
$$+ \sum_{n=a}^{b-2} x(n+1)p(n)x(n)w(n)\Big)$$
$$\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\Big(r^{2m+1}(b-1)|\Delta x(b-1)|^{2m+1}|x(b)|$$
$$+ r^{2m+1}(a)|\Delta x(a)|^{2m+1}|x(a)| + \sum_{n=a}^{b-2}|x(n+1)||p(n)||x(n)||w(n)|\Big).$$

Since

$$\begin{aligned}
|\Delta x(b-1)|^{2m+1} &= |x(b)-x(b-1)|^{2m+1}\\
&= |\sigma_2 x(b-1) - x(b-1)|^{2m+1} \le |\sigma_2 - 1|^{2m+1}M^{2m+1}\\
|x(b)| &= |\sigma_2 x(b-1)| \le \sigma_2 M\\
|\Delta x(a)|^{2m+1} &= |x(a+1)-x(a)|^{2m+1}\\
&= |x(a+1) - \sigma_1 x(a+1)|^{2m+1} \le |1-\sigma_1|^{2m+1}M^{2m+1}\\
|x(a)| &= |\sigma_1 x(a+1)| \le |\sigma_1|M,
\end{aligned}$$

substituting these facts to (7) we have

$$(2-|\sigma_1|-|\sigma_2|)^{2m+2}M^{2m+2}$$
$$\le \Big(\sum_{n=a}^{b-1}\frac{1}{r(n)}\Big)^{2m+1}\cdot\Big(r^{2m+1}(b-1)(1-\sigma_2)^{2m+1}|\sigma_2|M^{2m+2}$$
$$+ r^{2m+1}(a)(1-\sigma_1)^{2m+1}|\sigma_1|M^{2m+2}$$
$$+ \sum_{n=a}^{b-2}|p(n)|M^2(2M)^{2m}\sum_{i=1}^{2m+1} r^{2m+1-i}(n+1)r^{i-1}(n)\Big).$$

Dividing both sides by M^{2m+2} we get the desired inequality.

Remark 3. When $\sigma_1 = \sigma_2 = 0$, the inequality (6) reduced to the inequality (3).

References

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