TAMKANG JOURNAL OF MATHEMATICS
Volume 23, Number 3, Autumn 1992

CERTAIN CLASSES OF MEROMORPHIC MULTIVALENT FUNCTIONS

B. A. URALEGADDI AND C. SOMANATHA

Abstract. Let $f(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}\frac{a_{n-1}}{z^{p-n}}$ be regular in the punctured disk $E=\{z:0<|z|<1\}$ and $D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z)$ where $*$ denotes the Hadamard product and n is any integer greater than $-p$. For $-1\le B<A\le 1$, let $C_{n,p}(A,B)$ denote the class of functions $f(z)$ satisfying

$$-z^{p+1}(D^{n+p-1}f(z))'\prec p\frac{1+Az}{1+Bz},\qquad |z|<1.$$

This paper establishes the property $C_{n+1,p}(A,B)\subset C_{n,p}(A,B)$. Further property preserving integral operators, coefficient inequalities and a closure theorem for these classes are obtained. Our results generalise some of the recent results of Ganigi and Uralegaddi [1].

1. Introduction

Let $\sum_p$ denote the class of functions $f(z)=\frac{1}{z^p}+\frac{a_0}{z^{p-1}}+\frac{a_1}{z^{p-2}}+\cdots$, that are regular in the punctured disk $E=\{z:0<|z|<1\}$ and p a positive integer. Let $D^{n+p-1}f(z)=\frac{1}{z^p(1-z)^{n+p}}*f(z)$, where n is any integer greater than $-p$ and the operation $*$ denotes the Hadamard product. A function f of $\sum_p$ is said to be in the class $C_{n,p}(A,B)$ if

$$-z^{p+1}(D^{n+p-1}f(z))'\prec p\frac{1+Az}{1+Bz},\quad z\varepsilon\Delta=\{z:|z|<1\}\text{where}$$

Received July 20, 1991.
Keywords and phrases. Multivalent functions, Hadamard product, coefficient inequalities.
AMS 1980 subject classifications primary 30C45

$-1 \leq B < A \leq 1$, and the symbol $\prec$ denotes subordination.

Equivalently, a function f of $\sum_p$ belongs to $C_{n,p}(A,B)$ if and only if there exists a function w regular in Δ, satisfying $w(0) = 0$ and $|w(z)| < 1$ for $z\varepsilon\Delta$ such that

$$-z^{p+1}(D^{n+p-1}f(z))' = p\frac{1+Aw(z)}{1+Bw(z)} \tag{1}$$

It is easy to verify the identlty

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z). \tag{2}$$

Using (2), (1) may be written as

$$-z^p[(n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z)] = p\frac{1+Aw(z)}{1+Bw(z)} \tag{3}$$

This paper establishes the property $C_{n+1,p}(A,B) \subset C_{n,p}(A,B)$. Further we obtain class preserving integral operators, coefficient inequalities and a closure theorem for functions in these classes. By assigning speciffic values to A and B and putting $p = 1$, we obtain some of the results in [1, Th.1 and Th.2].

2. The classes $C_{n,p}(A,B)$.

We shall prove the following.

Lemma. *A function f in $\sum_p$ belongs to the class $C_{n,p}(A,B)$, $-1 \leq B < A \leq 1$ if and only if*

$$|z^{p+1}(D^{n+p-1}f(z))' + m| < M, \qquad z\varepsilon\Delta \tag{4}$$

where

$$m = p(1-AB)/(1-B^2) and M = p(A-B)/(1-B^2). \tag{5}$$

Proof is similar to that of lemma 2.1 in [3].

Theorem 1.

$$C_{n+1,p}(A,B) \subset C_{n,p}(A,B), \qquad n > -p.$$

Proof. Let $f \varepsilon C_{n+1,p}(A, B)$. Suppose that

$$-z^p[(n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z)] = p\frac{1+Aw(z)}{1+Bw(z)}.$$

That is

$$z^p[(n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z)] = -p\frac{1+Aw(z)}{1+Bw(z)}. \tag{6}$$

where w is either regular or meromorphic in Δ. clearly $w(0) = 0$ becasuce the left side of (6) is $-p$ when $z = 0$. Differentiating (6) and using (1) we obtain

$$z^{p+1}(D^{n+p}f(z))' = -p\frac{1+Aw(z)}{1+Bw(z)} - p\Big(\frac{A-B}{n+p}\Big)\Big[\frac{zw'(z)}{(1+Bw(z))^2}\Big].$$

Hence

$$\begin{aligned} z^{p+1}(D^{n+p}f(z))' + m &= \frac{(m-p)-(Ap-Bm)w(z)}{1+Bw(z)} \\ &\quad - p\Big(\frac{A-B}{n+p}\Big)\Big[\frac{zw'(z)}{(1+Bw(z))^2}\Big]. \end{aligned} \tag{7}$$

Let r^* be the distance from the origin to the nearest pole of w in Δ. Then w is reqular in $|z| < r_0 = \min\{r^*, 1\}$. By a lemma due to Jack [2] for $|z| < r(r < r_0)$ there exists a point z_0 such that

$$z_0 w'(z_0) = kw(z_0), \quad k \geq 1. \tag{8}$$

From (7) and (8) we have

$$z_0^{p+1}(D^{n+p}f(z_0))' + m = \frac{N(z_0)}{R(z_0)} \tag{9}$$

where

$$\begin{aligned} N(z_0) =&(n+p)(m-p) + [(n+p)(Bm-Ap) + B(n+p)(m-p) \\ &- kp(A-B)]w(z_0) + B(n+p)(Bm-Ap)W^2(z_0) \end{aligned}$$

and

$$R(z_0) = (n+p)(1+2Bw(z_0) + B^2W^2(z_0)].$$

Now suppose if possible $\max\limits_{|z|=r} |w(z)| = 1$ for some r, $r < r_0 \leq 1$. At the point z_0 where this occured, we would have $|w(z_0)| = 1$. Then by using the identities

$$p - m = BM \text{ and } Ap - Bm = M$$

we have

$$|N(z_0)|^2 - M^2|R(z_0)|^2 = x + 2y \quad \text{Re}\{W(z_0)\} \tag{10}$$

where

$$x = kp(A - B)\{kp(A - B) + 2M(n + p)(1 + B^2)\}$$

and

$$y = 2kp(A - B)MB(n + p).$$

From (10) we have

$$|N(z_0)|^2 - M^2|R(z_0)|^2 > 0, \quad \text{provided} \quad x \pm 2y > 0 \tag{11}$$

Now

$$x + 2y = kp(A - B)\{kp(A - B) + 2M(n + p)(1 + B)^2\} > 0$$

and

$$x - 2y = kp(A - B)\{kp(A - B) + 2M(n + p)(1 - B)^2\} > 0.$$

Thus it follows from (9) and (11) that

$$|z_0^{p+1}(D^{n+p}f(z_0))' + m| > M.$$

But, in view of the above lemma, this is a contradiction to the fact that $f\varepsilon C_{n+1,p}(A, B)$. So we cannot have $|w(z_0)| = 1$. Thus $|w(z)| \neq 1$ in $|z| < r_0$. Since $w(0) = 0$, $|w(z)|$ is continuous and $|w(z)| \neq 1$ in $|z| < r_0$, we cannot have a pole at $|z| = r_0$. Hence w is regular in Δ and satisfies $|w(z)| < 1$ for $z\varepsilon\Delta$. Therefore, from (6) it follows that $f\varepsilon Cn, p(A, B)$.

In the next theorem we obtain property preserving integral operators for $C_{n,p}(A, B)$.

Theorem 2. *Let p be a positive integer and n is any integer greater than $-p$. If $f(z)\varepsilon C_{n,p}(A,B)$ and $Re(c-p+1)>0$ then*

$$F(z) = \frac{c-p+1}{z^{c+1}}\int_0^z t^C f(t)dt\varepsilon C_{n,p}(A,B). \tag{12}$$

Proof. From the definition of F defined by (12) we have

$$z(D^{n+p-1}F(z))' = (c-p+1)D^{n+p-1}f(z)-(c+1)D^{n+p-1}F(z). \tag{13}$$

Let us suppose that

$$-z^{p+1}(D^{n+p-1}F(z))' = p\frac{1+Aw(z)}{1+Bw(z)} \tag{14}$$

where the function w is either regular or meromorphic in Δ. Clearly $w(0)=0$. Elimiating $(D^{n+p-1}F(z))'$ from (13) and (14), we have

$$(c-p+1)D^{n+p-1}f(z) = (c+1)D^{n+p-1}F(z)-pz^{-p}\Big[\frac{1+Aw(z)}{1+Bw(z)}\Big] \tag{15}$$

Differentiating (15) and using (14), we obtain

$$z^{p+1}(D^{n+p-1}f(z))' = -p\frac{1+Aw(z)}{1+Bw(z)}-p\Big(\frac{A-B}{c-p+1}\Big)\Big[\frac{zw'(z)}{(1+Bw(z))^2}\Big].$$

Hence

$$\begin{aligned} z^{p+1}(D^{n+p-1}f(z))'+m &= \frac{(m-p)-(Ap-Bm)w(z)}{1+Bw(z)} \\ &\quad -p\Big(\frac{A-B}{c-p+1}\Big)\Big[\frac{zw'(z)}{(1+Bw(z))^2}\Big]. \end{aligned} \tag{16}$$

The remaining part of the proof is similar to that of theorem 1.

Remark. For $p=1$, $A=1-2\alpha$ and $B=-1$ theorems 1 and 2 yield the earlier results in [1, Th.1 and Th.2].

Theorem 3. *Let p be a positive integer and n is any integer such that $n > -p$ and $F(z) = (n+p)z^{-n-2p}\int_0^z t^{n+2p-1}f(t)dt$. Then $F\varepsilon C_{n+1,p}(A,B)$ if and only if $f\varepsilon C_{n,p}(A,B)$.*

Proof. From the definition of F we have

$$(D^{n+p}F(z))' = (D^{n+p-1}f(z))'$$

and the result follows.

Now we obtain coefficient inequalities for the class $C_{n,p}(A,B)$.

Theorem 4. *Let $f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \frac{a_1}{z^{p-2}} + \cdots$. If $f\varepsilon C_{n,p}(A,B)$, then*

$$|a_{k-1}| \le \frac{p(A-B)}{(k-p)\alpha(n,k)}, \qquad k = 1,2,\ldots, \tag{17}$$

where $\alpha(n,j) = \binom{n+p+j-1}{n+p-1}$. Then result is sharp.

Proof. Since $f\varepsilon C_{n,p}(A,B)$, we have

$$(D^{n+p-1}f(z))' = -pz^{-p-1}\frac{1+Aw(z)}{1+Bw(z)}$$

where $w(z) = \sum_{j=1}^{\infty} t_j z^j$ is regular in Δ and $|w(z)| < 1$ for $z\varepsilon\Delta$. Then

$$(D^{n+p-1}f(z))' + pz^{-p-1} = -[Apz^{-p-1} + B(D^{n+p-1}f(z))']w(z)$$

or

$$\sum_{j=1}^{\infty}(j-p)\alpha(n,j)a_{j-1}z^{-p+j-1}$$
$$= -[(A-B)pz^{-p-1} + B\sum_{j=1}^{\infty}(j-p)\alpha(n,j)a_{j-1}z^{-p+j-1}]\sum_{j=1}^{\infty}t_j z^j. \tag{18}$$

Comparing coefficients of like powers of z on both the sides of (18) we see that the coefficient a_{k-1} on the left side of (18) depends only on $a_0, a_1, \ldots, a_{k-3}, a_{k-2}$

on the right side of (18). Hence for $j = 1, 2, \ldots$, it follows from (18) that

$$\sum_{j=1}^{k}(j-p)\alpha(n,j)a_{j-1}z^{-p+j-1} + \sum_{j=k}^{\infty} c_j z^{-p+j}$$
$$= -[(A-B)pz^{-p-1} + B\sum_{j=1}^{k-1}(j-p)\alpha(n,j)a_{j-1}z^{-p+j-1}]w(z)$$

where c_j are some coumplex numbers. Since $|w(z)| < 1$, by using parseval's identity we obtain

$$\sum_{j=1}^{k}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2 r^{2(-p+j-1)} + \sum_{j=k}^{\infty}|c_j|^2 r^{2(-p+j)}$$
$$\leq (A-B)^2p^2r^{2(-p-1)} + B^2\sum_{j=1}^{k-1}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2 r^{2(-p+j-1)}$$
$$\leq (A-B)^2p^2 + B^2\sum_{j=1}^{k-1}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2$$

Letting $r \to 1$ on the left side of this inequality we get

$$\sum_{j=1}^{k}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2 \leq (A-B)^2p^2 + B^2\sum_{j=1}^{k-1}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2$$

Thus

$$(k-p)^2(\alpha(n,k))^2|a_{k-1}|^2 \leq (A-B)^2p^2 - (1-B^2)\sum_{j=1}^{k-1}(j-p)^2(\alpha(n,j))^2|a_{j-1}|^2$$
$$\leq (A-B)^2p^2.$$

Hence

$$|a_{k-1}| \leq \frac{p(A-B)}{(k-p)\alpha(n,k)}.$$

The estimate is sharp for the function $f(z)$ given by

$$-z^{p+1}(D^{n+p-1}f(z))' = p\frac{1+Az^k}{1+Bz^k}, \qquad k = 1, 2 \ldots.$$

Further we obtain a sufficient coefficient condition, for a function to be in the class $C_{n,p}(A, B)$ when $-1 \leq B < 0$.

Theorem 5. *Let the function* $f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \frac{a_1}{z^{p-2}} + \ldots$ *be regular in* E *and* $-1 \leq B < 0$. *If*

$$\sum_{k=1}^{\infty}(k-p)(1-B)\alpha(n,k)|a_{k-1}| \leq p(A-B) \tag{19}$$

where $\alpha(n,k) = \binom{n+p+k-1}{n+p-1}$, *then* $f\varepsilon C_{n,p}(A,B)$. *The result is sharp.*

Proof. Suppose (19) holds. Then, for $z\varepsilon\Delta$, we have

$$\begin{aligned}
&|z^{p+1}(D^{n+p-1}f(z))' + p| - |Ap + Bz^{p+1}(D^{n+p-1}f(z))'| \\
&= |\sum_{k=1}^{\infty}(k-p)\alpha(n,k)a_{k-1}z^k| - |(A-B)p + B\sum_{k=1}^{\infty}(k-p)\alpha(n,k)a_{k-1}z^k| \\
&\leq \sum_{k=1}^{\infty}(k-p)\alpha(n,k)|a_{k-1}|r^k - \{(A-B)p + B\sum_{k=1}^{\infty}(k-p)\alpha(n,k)|a_{k-1}|r^k\} \\
&< \sum_{k=1}^{\infty}(k-p)\alpha(n,k)|a_{k-1}| - (A-B)p - B\sum_{k=1}^{\infty}(k-p)\alpha(n,k)|a_{k-1}| \\
&= \sum_{k=1}^{\infty}(k-p)(1-B)\alpha(n,k)|a_{k-1}| - (A-B)p \\
&\leq 0.
\end{aligned}$$

It follows that

$$|\{z^{p+1}(D^{n+p-1}f(z))' + p\}|/|\{Ap + Bz^{p+1}(D^{n+p-1}f(z))'\}| < 1 \tag{20}$$

It is easy to see that the inequality (20) is equivalent to (1). Hence $f\varepsilon C_{n,p}(A,B)$.

The estimate is sharp for the function

$$f(z) = \frac{1}{z^p} + \frac{P(A-B)}{(k-p)(1-B)\alpha(n,k)}z^{k-p}, \qquad k = 1, 2, \ldots.$$

For this function

$$|\{z^{p+1}(D^{n+p-1}f(z))' + p\}|/|\{Ap + Bz^{p+1}(D^{n+p-1}f(z))'\}| = 1, \text{ for } z = 1$$

and the equality is attained in (19).

Remark The converse of the above theorem need not be true. Consider the function

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \frac{a_1}{z^{p-2}} + \ldots \text{.where}$$

$$-z^{p+1}(D^{n+p-1}f(z))' = p\frac{1+Az}{1+Bz}, \quad -1 \le B < 0, \quad z\varepsilon\,\triangle\,.$$

Clearly $f\varepsilon C_{n,p}(A,B)$. Also it is easy to verify that

$$a_{k-1} = -\frac{(A-B)p(-B)^{k-1}}{(k-p)\alpha(n,k)}.$$

Hence

$$\sum_{k=1}^{\infty}(k-p)(1-B)\alpha(n,k)|a_{k-1}| = (A-B)p\sum_{k=1}^{\infty}(1-B)(-B)^{k-1} > (A-B)p.$$

The result follows.

Finally we state the following closure theorem for the class $C_{n,p}(A,B)$, the proof of which is obvious.

Theorem 6. *If the functions f and g belong to the class $C_{n,p}(A,B)$ and $0 \le s \le 1$, then the function F defind by $F(z) = sf(z)+(1-s)g(z)$ also belongs to $C_{n,p}(A,B)$.*

References

[1] M. D. Ganigi and B. A. Uralegaddi, "Subclasses of meromorphic close-to-convex functions," *Bull. Math. Soc. Sci. Math. R. S.* Roumanie (N.S.) 33(81) (1989), No. 2 105-109.

[2] I. S. Jack, "Functions Starlike and convex of order α," *J. London Math. Soc.* (2) 3 (1971), 469-474.

[3] Vinod Kumar and S. L. Shukla, "Multivalent functions defined by Ruscheweyh Derivatives-II," *Indian J. Pure appl. Math.* 15(11) (1984), 1228-1238.

Department of Mathematics, Karnatak University, Dharwad-580 003, India.