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OPERATORS ON BANACH ALGEBRA VALUED FUNCTION SPACES

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Abstract. Let S be a locally compact Hausdorff space and let A be a Banach algebra. Denote by $C_0(S,A)$ the Banach algebra of all A-valued continuous functions vanishing at infinity on S. Properties of bounded linear operators on $C_0(S,A)$, like multiplicativity, are characterized by Choy in terms of their representing measures. We study these theorems and give sharper results in certain cases.

1. Introduction

Let S be a locally compact Hausdorff space and let A be a Banach algebra. The Banach space (under the supremum norm) of all A-valued continuous functions vanishing at infinity on S will be denoted $C_0(S,A)$. When $A=\mathbb{C}$, we simply write it as $C_0(S)$. The dual and bidual of $C_0(S,A)$ are denoted by $C_0'(S,A)$ and $C_0''(S,A)$ respectively. Let $B(S)$ be the σ-algebra of all Borel subsets of S. To every bounded linear operator $T : C_0(S,A) \to C_0(S,A)$ there corresponds a finitely additive operator valued mearsure $m : B(S) \to L(A, C_0''(S,A))$ such that $Tf = \int_s f dm$ for all $f \in C_0(S,A)$. (See for example [1]). This is called the representing measure of T. If f is a complex function on S and if $x \in A$, $f \otimes x$ denote the function given by $f \otimes x(s) = f(s)x$ for every $s \in S$. When $e \subseteq S$, 1_e is the characteristic function of e. Let $e \in B(S)$ and $x \in A$, $1_e \otimes x$ can be

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viewed as an element in $C_0''(S,A)$ by $(1_e \otimes x)\mu = \int_e x d\mu$. Here $\mu \in C_0'(S,A)$ is identified as an A'-valued measure on S. We have $T''(1_e \otimes x) = m(e)x$. One may ask how properties of T are reflected in the representing measure m. This has been the subject matter of papers like [3] and [4] by Choy. Now $C_0(S,A)$ is a Banach algebra under the pointwise multiplication. The biduals A'' and $C_0''(S,A)$ are again Banach algebras under the Arens product as defined in Bonsall and Duncan [2, pp. 106-107]. The Arens product will play an important role in the subsequent discussion. However we shall refer readers to [2] and use the notations and properties given in the book without making any further reference.

The main result of [3] states that a bounded operator T is multiplicative, i.e. T satisfies $T(fg) = (Tf)(Tg)$ for all $f, g \in C_0(S,A)$, if and only if the representing measure m satisfies $m(e_1 \cap e_2)(xy) = (m(e_1)x)(m(e_2)y)$ for all $x, y \in A$ and $e_1, e_2 \in B(S)$. Writing $e_1 = (e_1 \setminus e_2) \cup (e_1 \cap e_2)$ and $e_2 = (e_2 \setminus e_1) \cup (e_1 \cap e_2)$, it can be easily verified that this latter condition is the same as

(1) $(m(e_1)x)(m(e_2)y) = 0$ whenever $e_1 \cap e_2 = \phi$, and

(2) $m(e)$ is multiplicative for all $e \in B(S)$.

In the next section we prove that a bounded operator T satisfies condition (1) above if and only if $(Tf)(Tg) = 0$ whenever $fg = 0$. We also prove that when A is unital, condition (2) alone ensures that T is multiplicative.

In the last section we turn to multiplicatively symmetric operators (defined below) and answer in negative a question raised in [4].

2. Multiplicative operators

Theorem 2.1. *The operator T has the property that $fg=0$ implies $(Tf)(Tg) = 0$ if and only if its representing measure m satisfies $(m(e_1)x)(m(e_2)y) = 0$ whenever $e_1 \cap e_2 = \phi$.*

Proof. (Necessity) Let $x, y \in A$ and let $e_1, e_2 \in B(S)$ such that $e_1 \cap e_2 = \phi$. For every $\mu \in C_0'(S,A)$,

$$(m(e_1)x)(m(e_2)y)(\mu) = T''(1_{e_1} \otimes x)T''(1_{e_2} \otimes y)(\mu) = \int_{e_1} x dT'[T''1_{e_2} \otimes y, \mu].$$

Let $\epsilon > 0$. Choose a compact subset $e_1' \subseteq e_1$, an open set $e_1'' \supseteq e_1'$ and an $f \in C_0(S)$ satisfying $0 \leq f \leq 1$, $f|_{e_1'} = 1$ and $f = 0$ outside e_1'' such that

$$|\int_{e_1} x dT'[T''1_{e_2} \otimes y, \mu] - \int_S f \otimes x dT'[T''1_{e_2} \otimes y, \mu]| < \epsilon.$$

Now $\int_S f \otimes x dT'[T''1_{e_2} \otimes y, \mu] = [T''1_{e_2} \otimes y, \mu](Tf \otimes x) = \int_{e_2} y dT' < \mu, Tf \otimes x >$. Take a compact subset $e_2' \subseteq e_2$, an open set $e_2'' \supseteq e_2'$ and a $g \in C_0(S)$ such that

$$|\int_{e_2} y dT' < \mu, Tf \otimes x > - \int_S g \otimes y dT' < \mu, Tf \otimes x > | < \epsilon.$$

We have

$$\int_S g \otimes y dT' < \mu, Tf \otimes x > = < \mu, Tf \otimes x > (Tg \otimes y) = \mu((Tf \otimes x)(Tg \otimes y)).$$

Now f and g can be chosen to satisfy $(f \otimes x)(g \otimes y) = 0$ and hence $\mu((Tf \otimes x)(Tg \otimes y)) = 0$. Since ϵ is arbitrary, we have $(m(e_1)x)(m(e_2)y) = 0$.

(Sufficiency) Suppose that $f, g \in C_0(S, A)$ satisfy $fg = 0$. For every $\epsilon > 0$, by considering the sets on which f, respectively g, are nonzero, and their relative complements from each other, we can find disjoint subsets $e_1, \cdots, e_l, e_{l+1}, \cdots, e_m, e_{m+1}, \cdots, e_n \in B(S)$ and $x_1, \cdots, x_m, y_{l+1}, \cdots, y_n \in A$ such that

$$\|f - \sum_{i=1}^{m} 1_{e_i} \otimes x_i\| < \epsilon, \ \|g - \sum_{j=l+1}^{n} 1_{e_j} \otimes y_j\| < \epsilon \quad \text{and}$$

$$(1_{e_i} \otimes x_i)(1_{e_j} \otimes y_j) = 0 \text{ for every } i = 1, \cdots, m \text{ and } j = l+1, \cdots, n.$$

Then $(m(e_i)x_i)(m(e_j)y_j) = 0$ and therefore

$$\begin{aligned} &\|TfTg\| \\ =&\|(\sum_{i=1}^{m} m(e_i)x_i + (Tf - \sum_{i=1}^{m} m(e_i)x_i))(\sum_{j=l+1}^{n} m(e_j)y_j + (Tg - \sum_{j=l+1}^{m} m(e_j)y_j))\| \\ <&\epsilon\|T\|^2(\|f\| + \|g\| + \epsilon). \end{aligned}$$

Since ϵ arbitrary, $TfTg = 0$.

Regarding the condition (2) described in the introduction, we first show that this property alone is not enough to guarantee that T is multiplicative.

Example 2.2. Let $S = \{s_1, s_2\}$ be a two point set and identify $C(S, A)$ with $A \times A$. We shall construct multiplicative operators φ, $\psi : A \to A$ such that $\varphi + \psi$ is multiplicative while $(\varphi x)(\varphi y) \neq 0$ for some $x, y \in A$. Then $T : C(S, A) \to C(S, A)$ defined by $T(x, y) = (\varphi(x) + \psi(y), 0)$ has representing measure m given by $m(\{s_1\})(x) = (\varphi(x), 0)$ and $m(\{s_2\})(x) = (\psi(x), 0)$. So m satisfies condition (2), but by Theorem 2.1, T is not multiplicative. Let A be $\mathbb{C}^3$ with a multiplication defined by $(a, b, c) \cdot (a', b', c') = (0, 0, ab' - ba')$. Under the usual l^2-norm, A is a Banach algebra. Let φ be the identity map on A and ψ be given by $\psi(a, b, c) = (0, a, 0)$. Then $\varphi + \psi$ is multiplicative and $\varphi(1, 0, 0)\psi(1, 0, 0) = (0, 0, 1)$.

When A is unital we have

Theorem 2.3. *If A is a unital Banach algebra, then $T : C_0(S, A) \to C_0(S, A)$ is multiplicative if and only if its representing measure m has the property that $m(e)$ is multiplicative for every $e \in B(s)$.*

Proof. We need only prove the sufficiency. Suppose that e_1 and e_2 are disjoint subsets in $B(S)$, we claim that $(m(e_1)x)(m(e_2)y) = 0$ for every $x, y \in A$. Let i be the unit element in A. Then $m(e_1)i, m(e_2)i$ and $m(e_1)i + m(e_2)i$ are all idempotents. It follows that $(m(e_1)i)(m(e_2)i) + (m(e_2)i)(m(e_1)i) = 0$. Multiplying both sides from the left by $(m(e_1)i)$ yields $(m(e_1)i)(m(e_2)i) + (m(e_1)i)(m(e_2)i)(m(e_1)i) = 0$. If we multiply the same element on the right, we get $(m(e_1)i)(m(e_2)i)(m(e_1)i) + (m(e_2)i)(m(e_i)i) = 0$. Hence $(m(e_1)i)(m(e_2)i) - (m(e_2)i)(m(e_1)i) = 0$. Thus we have $(m(e_1)i)(m(e_2)i) = 0$. So for every $x, y \in A$,

$$(m(e_1)x)(m(e_2)y) = (m(e_1)x)(m(e_1)i)(m(e_2)i)(m(e_2)y) = 0.$$

Theorem 2.4. *If A is a unital C^*-algebra or if A is a commutative C^*-algebra, then T on $C_0(S, A)$ is a $*$-algebra homomorphism if and only if $m(e)$ is* *a $*$-algebra homomorphism for every $e \in B(S)$.*

Proof. By [3, Theorem 3.3], T is involution preserving if and only if $m(e)$ is involution preserving for every $e \in B(S)$. Together with Theorem 2.3, the result follows in the unital case. In case A is a commutative C^*-algebra, it suffices to prove that if $m(e)$ is a $*$-algebra homomorphism for every $e \in B(S)$, $(m(e_1)x)(m(e_2)y) = 0$ whenever e_1 and e_2 are disjoint sets in $B(S)$ and $x, y \geq 0$. Using the fact that $m(e_1), m(e_2)$ and $m(e_1) + m(e_2)$ are multiplicative, we get $(m(e_1)x)(m(e_2)y) + (m(e_2)x)(m(e_1)y) = 0$. But from our assumption on m, the elements $m(e_1)x, m(e_2)y$ are positive. So are $(m(e_1)x)(m(e_2)y)$ and $(m(e_2)x)(m(e_1)y)$. It follows that $(m(e_1)x)(m(e_2)y) = 0$.

3. Multiplicatively symmetric operators

A bounded linear operator $T : C_0(S, A) \to C_0(S, A)$ is said to be multiplicatively symmetric if $T(fTg) = T((Tf)g)$ for every $f, g \in C_0(S, A)$. In [4, Theorem 2.2] Choy proved that a one-to-one linear operator T is multiplicatively symmetric if and only if $(1_{e_1} \otimes x)(m(e_2)y) = (m(e_1)x)(l_{e_2} \otimes y)$ for every $e_1, e_2 \in B(\text{supp} m)$ and $x, y \in A$. He asks whether the conclusion still holds if T is not assumed to be one-to-one. We show that the assumption is essential.

Example 3.1. Let A be $\mathbb{C}^2$ with the corrdinatewise multiplication and let S be any compact Hausdorff space. A function $f \in C(S, A)$ can be identified in the obvious way as an ordered pair (f_1, f_2) with $f_1, f_2 \in C(S)$. Define T by $T(f_1, f_2) = (0, f_1)$. Then T is multiplicatively symmetric but not one-to-one. We have $\text{supp}(m) = S$. But if we let $e_1 = e_2 = S$, $x = (1, 2)$ and $y = (2, 1)$, Then $(1_{e_1} \otimes x)(m(e_2)y) = (0, 4) \neq (0, 1) = (m(e_1)x)(1_{e_2} \otimes y)$.

It may be desirable to have a description of multiplicatively symmetric operators in terms of their representing measures, but we have not been able to obtain a reasonably simple formula.

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