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NEW REPRESENTATION OF A SPECIAL NON-SYMMETRIC HOMOGENEOUS DOMAIN IN $\mathbb{C}^n$ $n \geq 6$

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1. Introduction

Let D be a homogeneous bounded domain in $\mathbb{C}^n$, $n \geq 6$. One of the problems for the homogeneous bounded domains in $\mathbb{C}^n$, $n \geq 3$ is to classify those which are not symetric. In each $\mathbb{C}^4$ and $\mathbb{C}^5$ there is only one non-symmetric homogeneous bounded domain. In a previous paper we have given a new representation of each those non-symmetric homogeneous bounded domains in $\mathbb{C}^4$ and $\mathbb{C}^5$ ([10]).

The aim of this paper is to describe with a new representation one of the non-symmetric homogeneous bounded domains in $\mathbb{C}^n$, where $n \geq 6$.

The whole paper contains three paragraphs. Each of them is analyzed as follows.

In the second paragraph we give the relation between these domains, Siegel domains and normal J-algebras.

The description of this special non-symmetric homogeneous bounded domain in $\mathbb{C}^n$, $n \geq 6$, is included in the last paragraph.

2. Let $\mathbb{C}^n$ be the n dimensional Euclidean complex space.

An open connected subset D of the $\mathbb{C}^n$ is called domain. We denote by $G(D)$ the group of all holomorphic automorphisms of D. If D is bounded, then $G(D)$

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is a Lie group are there exists on D a volume element w, which is defined by

$$w = (\sqrt{-1})^{n^2} k dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \ldots d\bar{z}_n$$

where $z_1, \ldots, z_n$ are complex coordinates in $\mathbb{C}^n$ and k the Bergaman function on D, which is positive. The Bergman function k gives a Kähler metric g on D defined by

$$g = \sum_{h=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^2 \log k}{\partial z_h \partial \bar{z}_\ell} dz_h \wedge d\bar{z}_\ell$$

and therefore (D, g) is a Kähler manifold. The bounded domain D in $\mathbb{C}^n$ is called homogeneous, if the group $G(D)$ acts transitively on D and therefore D, in this case, can be written

$$D = G(D)/H, \tag{2.1}$$

where H is the isotropy subgroup of $G(D)$ at the point $z_0 \in \mathbb{C}^n$. The relation (2.1) can also be written as follows

$$d = G_0(D)/H_0$$

where $G_0(D)$ is the identiy component of $G(D)$ and H_0 the isotropy subgroup of $G_0(D)$ at $z_0 \in D$

It is known that there exists a solvable Lie subgroup S of $G(D)$ which can be identified with D

Therefore S is a kähler manifold on which there exists a complex structure on it denoted by $\mathcal{J}$.

Let s be the Lie algebra of S which can be identified with the tangent space of S at its identity element e. The almost complex structure J on D defines an endomorphism $\mathcal{J}_0$ on s with the following properties

$$\mathcal{J}_0 : s \to s, \mathcal{J}_0 : X \to \mathcal{J}_0(x), \quad \mathcal{J}_0^2 = -id \tag{2.2}$$

This endomorphism $\mathcal{J}_0$ satisfies the following relation

$$[X,Y] + \mathcal{J}_0([\mathcal{J}_0(X), Y]) + \mathcal{J}_0([X, \mathcal{J}_0(Y)]) - [\mathcal{J}_0(X), \mathcal{J}_0(Y)] = 0 \tag{2.3}$$

which is obtained from the fact that the almost complex structure on D is integrable. The Kähler metric g on D induces a Hermitian positive definite symmetric bilinear form B on s. From B we obtain a quadratic form ω defined by

$$\omega : s \to R, \quad \omega : X \to \omega(X) \; = \; B(X, \mathcal{J}_0(X)) \tag{2.4}$$

satisfying the following conditions

$$\omega([\mathcal{J}_0(X), \mathcal{J}_0(Y)]) \; = \; \omega([X, Y]) \tag{2.5}$$

$$\omega([\mathcal{J}_0(X), X]) > 0 \qquad X \neq 0 \tag{2.6}$$

Therefore from the homogeneous bounded domain $D = G/H$ we obtain the set $\{s, \mathcal{J}_0, \omega\}$, where s a special solvable Lie algebra, $\mathcal{J}_0$ is an endomorphism on s having the properties (2.2) and (2.3) and ω quadratic form on s with the properties (2.5) and (2.6)

This set $\{s, \mathcal{J}_0, \omega\}$ is called normal J-algebra

Every normal J-algebra has also the property that the operator

$$\alpha d\tau_0 \;:\; s \to s, \; \alpha d\tau_0 \;:\; \tau \to \alpha d\tau_0(\tau) \; = \; [\tau_0, \tau] \tag{2.7}$$

has only real characteristic roots $\forall \tau_0 \in s$, that is, $\alpha d\tau_0$, as a matrix, is $\mathbb{R}-$triangular

The inverse is also true. Let $(s, \mathcal{J}_0, \omega)$ be a triple, where s is a solvable Lie algebra having the property (2.7), $\mathcal{J}_0$ an endomorphism on s having the properties (2.2) and (2.3) and ω quadratic form on s having the properties (2.5) and (2.6). Then there exists a unique solvable Lie group S whose Lie algebra is s which can be identified with the tangent space of S at its identity e. The endomorphism $\mathcal{J}_0$ on s gives arise the complex structure on s and finally the quadratic form ω on s induces a Hermitian inner product on s defined by

$$< X, Y > \; = \; \omega([\mathcal{J}_0 \, X, Y]) \tag{2.8}$$

which determines the Kähler metric g on S. The couple (S, g) is a Kähler manifold beholomorphically isomorphic onto homgeneous bounded domain in

$\mathbb{C}^n$. In the next paragraph we shall give one triplet $(s, \mathcal{J}_0, \omega)$ and the Kähler manifold (S, g) which is obtained by this triple.

3. We consider the solvable Lie algebra s, which can be described by the set of matrices

$$s = \left\{ A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 \cdots x_{n-2} & x_{n-1} & x_n \\ 0 & \phi_1 & 0 & 0 & 0 \cdots 0 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 & 0 \cdots 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_2 & 0 \cdots 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \cdots \phi_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & \phi_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & \phi_n \end{bmatrix} \begin{matrix} x_1, x_2 \cdots, x_1 \in \mathbb{R}^* \\ \phi_1, \phi_2, \cdots, \phi_n \in \mathbb{R}^* \end{matrix} \right\} \tag{3.1}$$

From this constuction of s we conclude that the endomorphism $\mathcal{J}_0$ has the form

$$\mathcal{J}_0 = (\beta_{k\ell}), \ \beta_{k\ell} \in \mathbb{R} \quad k = 1, \ldots, n, \ \ell = 1, \ldots, n \tag{3.2}$$

which must satisfy the relations (2.2) and (2.3)

From these conditions and after a lot of estimates we obtain

$$\mathcal{J}_0 = \begin{bmatrix} p_1 & 0 & 0 \cdots 0 & \xi_1 & 0 & 0 \cdots 0 \\ 0 & p_2 & 0 \cdots 0 & 0 & \xi_2 & 0 \cdots 0 \\ 0 & 0 & p_3 \cdots 0 & 0 & 0 & \xi_3 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots p_n & 0 & 0 & 0 \cdots \xi_n \\ -\frac{1+p_1^2}{\xi_1} & 0 & 0 \cdots 0 & -p_1 & 0 & 0 \cdots 0 \\ 0 & -\frac{1+p_1^2}{\xi_2} & 0 \cdots 0 & 0 & -p_2 & 0 \cdots 0 \\ 0 & 0 & -\frac{1+p_3^2}{\xi_3} \cdots 0 & 0 & 0 & -p_3 \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots -\frac{1+p_n^2}{\xi_n} & 0 & 0 & 0 \cdots - p_n \end{bmatrix} \begin{matrix} , p_1, \cdots, p_n \in \mathbb{R} \\ \xi_1, \cdots, \xi_n \in \mathbb{R}^* \end{matrix} \tag{3.4}$$

The quadratic form ω, on this Lie algebra s, is difined by

$$\omega(X) = < X_0, X > \tag{3.5}$$

where $< >$ the usual inner product on s and $X_0 = (K_1, K_2, \ldots, K_n, K_{n+1}, \ldots, K_{2n})$ is a fixed vector. In order that ω satisfies the conditions (2.5) and (2.6) we must have

$$K_1\xi_1 > 0, \quad K_2\xi_2 > 0, \ldots, \quad K_n\xi_n > 0 \tag{3.6}$$

Now, we have proved the following theorem

Theorem 3.1. *There exists a homogeneous bounded domain in* $\mathbb{C}^n$, $n \geq 6$ *haveing* $(s, \mathcal{J}_0, \omega)$ *normal J-algebra, where* s, $\mathcal{J}_0$ *and* ω *are given by* (3.1), (3.4) *and* (3.5) *respectively*

Now, we determine the solvable Lie group S which corresponds to the solvable Lie apgebra s

We denote by $GL(s)$ the group of all nonsigular endomorphisms of s. The Lie algebra $gl(s)$ of $GL(s)$ consists of all endomorphisms of s with the standard bracket operation

$$[X, Y] = XY - YX \tag{3.7}$$

The mapping

$$\alpha d : s \to gl(s), \qquad \alpha d : B \to \alpha dB \tag{3.8}$$

where

$$\alpha dB : s \to s, \qquad \alpha dB : T \to \alpha dB(T) = [T, B] \tag{3.9}$$

is a homomorphism of s onto a subalgebta $\alpha d(s)$ of $gl(s)$. Let Int(s) be the analytic subgroup of $GL(s)$ whose Lie algebra is $\alpha d(s)$ which is called adjoint group of s. The group Aut(s) of all automorphisms of s is a closed subgroup of $GL(s)$. Thus Aut(s) has a unique analytic structure under which it becomes a topological Lie subroup of $GL(s)$. We denote by $d(s)$ the Lie algebra of Aut(s). Now, the group Int(s) is connected, so it is generated by elements $e^{\alpha dX}$, $X \in s$. Therefore Int(s) is a normal subgroup of Aut(s)

From the above we conclude that the solvable Lie group S of s is defined

$$s=\left\{ L=\begin{bmatrix} 1 & \frac{x_1}{\psi_1}(e^{\psi_1}-1) & \frac{x_2}{\psi_2}(e^{\psi_2}-1) & \cdots & \frac{s_n}{\psi_n}(e^{\psi}-1) \\ 0 & e^{\psi_1} & 0 & \cdots & 0 \\ 0 & 0 & e^{\psi_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & e^{\psi_n} \end{bmatrix} \begin{matrix} x_i \in \mathbb{R} \\ y_i \in \mathbb{R} \\ i=1,\ldots,n \end{matrix} \right\} \tag{3.10}$$

The inner product on the solvable Lie algebra is defined by

$$< X, Y > = \omega([\mathcal{J}_0 \, X, Y]) \tag{3.11}$$

where ω is given by (3.5). This inner product determines the Kähler metric on S which is essentially the Bergaman metric on it.

Now, we can state the following theorem

Theorem 3.2. *The homogeneous non-symmetric bounded domain in* $\mathbb{C}^n$ $n \geq 6$ *is biholomorphically isomorphic onto the solvable Lie group defined by* (3.10). *The Kähler metric* g *on* S *defined by the relation* (3.11).

Let F be a Lie automorphism on s. This F can be represented by matrix

$$F = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0, & a_{1,n+1} & 0 & 0 & \cdots 0 \\ 0 & a_{22} & 0 & \cdots & 0 & 0 & a_{2,n+2} & 0 & \cdots 0 \\ 0 & 0 & a_{33} & \cdots & 0 & 0 & 0 & a_{3,n+3} & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} & 0 & 0 & 0 & \cdots a_{n,2n} \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots 1 \end{bmatrix}$$

which becomes an isometry with respect to the inner product

$$< X, Y > = < X, [\mathcal{J}_0 X, Y] > = \omega([X, Y])$$

If we have

$$a_{\ell\ell} = \pm 1, \quad \ell = 1, 2, \ldots, n \quad a_{\ell,\ell+n} = \frac{p_\ell \xi_\ell}{1+p_\ell^2} \cdot \frac{1-a_{\ell\ell}}{a_{\ell\ell}} \quad \ell = 1, \ldots, n$$

$$F_{isom} = \begin{bmatrix} a_{11} & 0 & 0\cdots 0 & \frac{p_1\xi_1}{1+p_1^2}\frac{1-a_{11}}{a_{11}} & 0 & 0 & \cdots 0 \\ 0 & a_{22} & 0\cdots 0 & 0 & \frac{p_2\xi_2}{1+p_2^2}\frac{1-a_{22}}{a_{22}} & 0 & \cdots 0 \\ 0 & 0 & a_{33}\cdots 0 & 0 & 0 & \frac{p_3\xi_3}{1+p_3^2}\frac{1-a_{33}}{a_{33}} & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots a_{nn} & 0 & 0 & 0 & \frac{p_n\xi_n}{1+p_n^2}\frac{1-a_{nn}}{a_{nn}} \\ 0 & 0 & 0\cdots 0 & 1 & 0 & 0 & \cdots 0 \\ 0 & 0 & 0\cdots 0 & 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 0\cdots 0 & 0 & 0 & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0\cdots 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From the form F_{isom} we obtain that it has the eigenvalue 1 with multiplicity n.

Therofore we have proved the following theorem.

Theorem 3.3. *The homogeneous bounded domain in* $\mathbb{C}^n$ $n \geq 6$ *described by the theorem* 3.2 *does not admit any k-symmetric structure.*

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