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RINGS WITH (x, R, x) AND $(N + NR, R)$ IN THE LEFT NUCLEUS

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Abstract. Let R be a nonassociative ring, N the left nucleus. Assume that N is a nonzero Lie ideal of R. It is shown that if R is a prime ring which satisfies $(x, R, x) \subset N$ and $(NR, R) \subset N$ then R is either associative or commutative.

1. Introduction

Kleinfeld [1] weakened Thedy's hypotheses [2] to obtain the following result: If R is a prime nonassociative ring which satisfies $(x, R, x) \subset N$ and $(R, R) \subset N$, then R is either associative or commutative. In [4], we weaken Kleinfeld's hypotheses to obtain the same result. In this note, we weaken Kleinfeld's second hypothesis to obtain this result.

2. Main result

Let R be a nonassociative ring. We adopt the usual notation for associators and commutators: $(x, y, z) = (xy)z - x(yz)$, $(x, y) = xy - yx$. We shall denote the left nucleus by N. Thus N consists of all elements n such that $(n, R, R) = 0$, where we assume that R is a ring in which (I) $(x, R, x) \subset N$ and (II) $(N + NR, R) \subset N$. Using (II), we obtain

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$$(RN, R) \subset ((R, N) + NR, R) \subset N. \tag{1}$$

A linearization of (I) yields

$$(x, y, z) + (z, y, x) \in N. \tag{2}$$

In every ring one may verify the identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) \ = \ w(x, y, z) + (w, x, y)z. \tag{3}$$

Definition. Let $J(x, y, z) = (x, y, z) + (y, z, x) + (z, x, y)$.

In every ring we have the identity

$$(xy, z) + (yz, x) + (zx, y) = J(x, y, z). \tag{4}$$

Consequently, using (II) and (1) we have

$$J(x, y, z) \in N \quad \text{if one of} \quad x, y \quad \text{and} \quad z \quad \text{is in} \quad N. \tag{5}$$

Moreover, in every ring we have the identity

$$(xy, z) = x(y, z) + (x, z)y + J(x, y, z) - (x, z, y) - (y, z, x). \tag{6}$$

Then combining (6), (2), (5) and (II) we obtain

$$x(y, z) + (x, z)y \in N \quad \text{for all} \quad x \quad \text{in} \quad N. \tag{7}$$

Suppose that $n \in N$. Then with $w = n$ in (3) we have $(nx, y, z) = n(x, y, z)$. Combining this with (II) yields

$$(nx, y, z) = n(x, y, z) = (xn, y, z) \quad \text{for all} \quad n \quad \text{in} \quad N. \tag{8}$$

As a consequence of (8), (7), (1) and (II), we have that N is an associative subring of R, and thus

$$(N,R)(NR) \subset N \quad \text{and} \quad (N,R)(RN) \subset N. \tag{9}$$

Definition. Let $T = \{t \in N : t(R,R,R) = 0\}$.

Using (9), (8) and (II), we have that $((N,R)(NR))(R,R,R) =$ $(((N,R)(NR))R,R,R) = ((N,R)((NR)R)\ ,R,R) = ((N,R)(NR^2),R,R) = 0.$ Hence we obtain

$$(N,R)(NR) \subset T. \tag{10}$$

Lemma 1. *T is an ideal of R and $T(R,R,R) = 0$.*

Proof. Substitute t for n in (8). Then $(tx,y,z) = t(x,y,z) = (xt,y,z) = 0$. Thus $tR \subset N$ and $Rt \subset N$. First note that $tw \cdot (x,y,z) = t \cdot w(x,y,z)$. But (3) multiplied on the left by t yields $t \cdot w(w,y,z) = -t \cdot (w,x,y)z = -t(w,x,y) \cdot z = 0$. Hence $tw \cdot (x,y,z) = 0$. On the other hand, using $tR \subset T$, (8), (II), (1) and (2) we obtain $wt \cdot (x,y,z) = (w,t)(x,y,z) = ((w,t)x,y,z) = ((wt)x,y,z) - (t(wx),y,z) = ((wt,x),y,z) + (x(wt),y,z) = -((x,w,t),y,z) + ((xw)t,y,z) = -((x,w,t) + (t,w,x),y,z) + ((xw,t),y,z) + (t(xw),y,z) = 0$.At this point we have verified that T is an ideal of R. The rest is obvious. This completes the proof of the lemma.

Definition. Let I be the associator ideal of R.

I consists of the smallest ideal which contains all associators.

Note that I may be characterized as all finite sums of associators and right multiples of associators, as a consequence of (3).

Henceforth assume not only that R satisfies (I) and (II), but also that R is semiprime. By that is meant that the only ideal of R which squares to zero is the zero ideal. Using Lemma 1 and (3) we have that $T \cdot I = 0$ and so $(T \cap I)^2 = 0$. Thus we obtain

(11) $T \cap I = 0$.

Lemma 2. $(R,R,N)=0$.

Proof. Assume that $n\in N$. Using (2) we get $(x,y,n)=(x,y,n)+(n,y,x)\in N$. Also (3) implies $z(x,y,n)=(zx,y,n)-(z,xy,n)+(z,x,yn)-(z,x,y)n$. Hence using (8) and (2) we obtain $(x,y,n)(z,r,s)=(z(x,y,n),r,s)$ $=((z,x,yn),r,s)-((z,x,y)n,r,x)=-((yn,x,z),r,s)-n((z,x,y),r,s)=$ $-(n(y,x,z),r,s)-((z,x,y),r,s)=-n((y,x,z)+(z,x,y),r,s)=0$, so that $(x,y,n)\in T$. Since this element is also an associator, it obviously is also in I. Thus by (11), $(x,y,n)=0$, as desired.

Using Lemma 2 and (II) we obtain

(12) $n\in N$ and $(n,R)=0$ imply $n(R,R)\subset(NR,R)\subset N$.

Recall that a ring is called prime if the product of any two nonzero ideals is nonzero. We have our

Main Theorem. *If R is a prime nonassociative ring with $N\neq 0$ satisfying (I) and (II), then R is either associative or commutative.*

Proof. Since $T\cdot I=0$, we have $I=0$ or $T=0$. If $I=0$, then R is associative. Assume that $T=0$. Using (10) and (II), we obtain $((N,R)N)R=(N,R)(NR)=0$. Thus, $((N,R)N)(R,R,R)=0$ and so $(N,R)N\subset T$. Hence $(N,R)N=0$. So by (II), $(N,R)(N,R)=0$ and thus $(N,R)(RN)=0$. Using (II), we see that $R(N,R)\subset(R,(N,R))+(N,R)R\subset(N,R)+(N,R)R$ and so the ideal generated by (N,R) is $(N,R)+(N,R)R$ by Lemma 2. Hence, it is easy to show that $((N,R)+(N,R)R)^2=0$. This implies $(N,R)+(N,R)R=0$ by primeness of R. So, $(N,R)=0$. By Lemma 2, $(N,R,R)=(R,R,N)=0$. Thus NR is a nonzero ideal of R. Let $K=\{x\in R: Nx=0\}$. Then K is an ideal of R and $NK=0$. So, $(NR)K=N(RK)\subset NK=0$. Hence $K=0$ by primeness of R. Using (8) and (12), we obtain $N((R,R),R,R)=(N(R,R),R,R)=0$. Thus $((R,R),R,R)\subset K$ and so $((R,R),R,R)=0$. Hence $(R,R)\subset N$. It follows from this and (I) that R satisfies Kleinfeld's hypotheses [1] and thus the conclusion is valid. This completes the proof of the Main Theorem.

Thedy's example [3] shows that Kleinfeld's hypothesis $(R,R)\subset N$ can not be replaced by the weaker condition $(N, R) \subset N$. It is interesting to ask whether this hypothesis can be replaced by $((R, R), R, R) \subset N$.

Note added in Proof. Using the result of [4], we can improve the Main Theorem as follows: If R is a prime ring with $N \neq 0$ satisfying $(x, y, z)+(z, y, x) \in N$ and $(N + NR, R) \subset N$, then R is either associative or commutative.

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