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ON COMMUTATIVITY OF ONE-SIDED s-UNITAL RINGS

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Abstract. In the present paper, we study the commutativity of one sided s-unital rings satisfying conditions of the form $[x^r y \pm x^n y^m x^s, x] = 0 = [x^r y^m \pm x^n y^{m^2} x^s, x]$, or $[yx^r \pm x^n y^m x^s, x] = 0 = [y^m x^r \pm x^n y^{m^2} x^s, x]$ for each $x, y \in R$, where $m = m(y) > 1$ is an integer depending on y and n, r and s are fixed non-negative integers. Other commutativity theorems are also obtained. Our results generalize some of the well-known commutativity theorems for rings.

Throughout the present paper, R will represent an associative ring (not necessarily with unity 1). Let $Z(R)$ denotes the center of R, $N(R)$ the set of all nilpotent elements of R, $N'(R)$ the set of all zero-divisors of R and $C(R)$ the commutator ideal of R. By $(GF(q))_2$ we mean the ring of 2×2 matrices over the Galois field $GF(q)$ with q elements. As usual $\mathbb{Z}[t]$ is the totality of polynomials in t with coefficients in $\mathbb{Z}$, the ring of integers, and for each $x, y \in R$, $[x, y] = xy - yx$.

A ring R is called left (resp. right) s-unital if $x \in Rx$ (resp. $x \in xR$) for every $x \in R$. Further, R is called s-unital if $x \in Rx \cap xR$ for all $x \in R$. If R is s-unital (resp. left s-unital or right s-unital), then for any finite subset F of R there exists an element $e \in R$ such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$) for all $x \in F$. Such an element e will be called a pseudo (resp. pseudo left or pseudo right) identity of F in R (see [16, 20, 21]).

In [9] it was studied the following ring properties:

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(P_1) : For each x, y in R, $[x, x^r y - x^n y^m x^s] = 0$ where $r \geq 1$, $n \geq 1$, $m > 1$, $s \geq 1$ are fixed non-negative integers.

$(P_1)^*$: For each x, y in R, $[x, yx^r - x^n y^m x^s] = 0$ where $r \geq 1$, $n \geq 1$, $m > 1$, $s \geq 1$ are fixed non-negative integers.

(P_2) : For each y in R, there exists integer $m = m(y) > 1$ such that $[x, xy - x^n y^m x^s] = 0 = [x, xy^m - x^n y^{m^2} x^s]$ for all x in R, where n, s are fixed integers.

$(P_2)^*$: For each y in R, there exists integer $m = m(y) > 1$ such that $[x, yx - x^n y^m x^s] = 0 = [x, y^m x - x^n y^{m^2} x^s]$ for all x in R, where n, s are fixed integers.

Indeed it was proved the following results:

Theorem A_1. *If R is a ring with unity 1 satisfies either of the properties (P_1) or $(P_1)^*$, then R is commutative.*

Theorem A_2. *Let R be a ring with unity 1 satisfying either of the properties (P_2) or $(P_2)^*$. Then R is commutative.*

Further, in [9] the above results were extended to a class of rings called one-sided s-unital rings. Actually it was proved the following:

Theorem A_3. *Let R be a left s-unital ring satisfying (P_2). Then R is commutative.*

Theorem A_4. *Let R be a right s-unital ring satisfying $(P_2)^*$. Then R is commutative.*

The aim of the present paper is to generalize the above mentioned results and the results proved in [1]-[10]. Also, correct some of the results in [5]. In fact we prove the following:

Theorem 1. *Let $m = m(y) > 1$ be an integer depending on y and n, r and s be fixed non-negative integers. If R is a left s-unital ring which satisfies the* *polynomial identity*

$$[x^r y \pm x^n y^m x^s, x] = 0 \text{ for all } x, y \in R, \tag{1}$$

$$[x^r y^m \pm x^n y^{m^2} x^s, x] = 0 \text{ for all } x, y \in R, \tag{1'}$$

then R is commutative.

Theorem 2. *Let $m = m(y) > 1$ be an integer depending on y and n, r and s be fixed non-negative integers. If R is a right s-unital ring which satisfies the polynomial identity*

$$[yx^r \pm x^n y^m x^s, x] = 0 \text{ for all } x, y \in R, \tag{2}$$

$$[y^m x^r \pm x^n y^{m^2} x^s, x] = 0 \text{ for all } x, y \in R, \tag{2'}$$

then R is commutative.

In preparation for the proof of our results, we need the following well-known results:

Lemma 1 ([15, Lemma 3]). *Let R be a ring such that $[x, [x, y]] = 0$ for all $x, y \in R$. Then $[x^m, y] = mx^{m-1}[x, y]$ for any positive integer m.*

Lemma 2 [18, Lemma 1]. *Let R be a ring with unity 1. If for each $x, y \in R$, there exists an integer $k = k(x, y) \geq 1$ such that $x^k[x, y] = 0$ or $[x, y]x^k = 0$, then $[x, y] = 0$.*

Lemma 3 ([17, Lemma 3]). *Let R be a ring with unity 1. If $(1 - y^n)x = 0$, then $(1 - y^{nm})x = 0$ for any positive integer m.*

Lemma 4 ([13, Theorem]). *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \ldots, x_n$ with integer coefficients. Then the following statements are equivalent:*

(1) *For any ring R satisfying $f = 0$, $C(R)$ is a nil ideal.*

(2) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

(3) *Every semi-prime ring satisfying* $f = 0$ *is commutative.*

Lemma 5 ([23, Lemma 1]). *Let* R *be a ring with unity 1, and let* $I_0^r(x) = x^r$ *for all* $x \in R$. *Define* $I_k^r(x)$ *inductively by* $I_k^r(x) = I_{k-1}^r(x+1) - I_{k-1}^r(x)$ *for all positive integers* k. *Then for all* $x \in R$, *we have* $I_{r-1}^r(x) = (r-1)r!/2 + r!x$, $I_r^r(x) = r!$ *and* $I_j^r(x) = 0$ *for all* $j > r$.

Lemma 6 ([22 , Lemma]). *Let* R *be a left (resp. right) s-unital ring. If for each pair of elements* x *and* y *in* R, *there exists a positive integer* $k = k(x,y)$ *and an element* $e = e(x,y)$ *of* R *such that* $x^k e = x^k$ *and* $y^k e = y^k$ *(resp.* $ex^k = x^k$ *and* $ey^k = y^k$*), then* R *is s-unital.*

Next, we consider the following ring property:

(H) For each x, y in R there exists $f(t) \in t^2\mathbb{Z}[t]$ such that $[x - f(x), y] = 0$.

Theorem *H* ([11, Theorem]). *Every ring satisfying (H) is commutative.*

In order to prove Theorem 1, we establish two lemmas.

Lemma 7. *Let* $m = m(x,y) > 1$, $n = n(x,y)$, $r = r(x,y)$ *and* $s = s(x,y)$ *be non-negative integers and let* R *be a left s-unital ring satisfying* $[x^r y \pm x^n y^m x^s, x] = 0$ *for all* $x, y \in R$. *Then* R *is an s-unital ring.*

Proof. If $x, y \in R$, then there exists $e = e(x,y) \in R$ such that $ex = x$ and $ey = y$. Further, there exist integers $m = m(x,e) > 1$, $n = n(x,e)$, $r = r(x,e)$, and $s = s(x,e) > 0$ such that $x^r[x,e] = \pm x^n[x,e^m]x^s$. So $x^{r+1}e - x^r ex = \pm(x^{n+1}ex^s - x^n ex^{s+1})$. Thus $x^{r+1}e = x^{r+1}$. Also, if $m_1 = m(y,e) > 1$, $n_1 = n(y,e)$, $r_1 = r(y,e)$ and $s_1 = s(y,e) > 0$, then we get $y^{r_1+1}e = y^{r_1+1}$. Thus $x^{r+r_1+2}e = x^{r+1}(x^{r_1+1}e) = x^{r+r_1+2}$ and $y^{r+r_1+2}e = y^{r+r_1+2}$. Therefore, R is s-unital by Lemma 6. If $s = 0$, then $x^{r+1}y - x^r yx = \pm(x^{n+1}y^m - x^n y^m x)$. So $e^{r+1}y - e^r ye = \pm(e^{n+1}y^m - e^n y^m e)$ and thus $y = ye \pm (y^m - y^m e) = y(e \pm (y^{m-1} - y^{m-1}e)) \in yR$. Therefore, R is s-unital.

Lemma 8. *Let* $m = m(x,y) > 1$, $n = n(x,y)$, $r = r(x,y)$ *and* $s = s(x,y)$ *be non-negative integers. If* R *satisfies* $[x^r y \pm x^n y^m x^s, x] = 0$ *for all* $x, y \in R$, *then $C(R) \subseteq N(R)$. Further, if R has unity 1, then $C(R) \subseteq Z(R)$.*

Proof. By the hypothesis, we have

$$x^r[x,y] = \pm x^n[x,y^m]x^s \text{ for all } x,y \in R. \tag{3}$$

Let $x = e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $y = e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $(GF(p))_2$ for any prime p. Then x and y fail to satisfy (3). By Lemma 4, $C(R) \subseteq N(R)$.

If $m_1 = m(x,y) > 1$, $n_1 = n(x,y)$, $r_1 = r(x,y)$, and $s_1 = s(x,y)$, then

$$x^{r_1}[x,y] = x^{n_1}[x,y^{m_1}]x^{s_1} \quad \text{for all } x,y \in R,$$

or

$$x^{r_1}[x,y] = x^{n_1}[y^{m_1},x]x^{s_1} \quad \text{for all } x,y \in R.$$

Now, let $m_2 = m(x,y^{m_1}) > 1$, $n_2 = n(x,y^{m_1})$, $r_2 = r(x,y^{m_1})$, and $s_2 = s(x,y^{m_1})$. Then

$$\begin{aligned} x^{r_1+r_2}[x,y] &= x^{r_2}(x^{n_1}[x,y^{m_1}]x^{s_1}) \\ &= x^{n_1+n_2}[x,y^{m_1m_2}]x^{s_1+s_2}, \end{aligned}$$

or

$$\begin{aligned} x^{r_1+r_2}[x,y] &= x^{r_2}(x^{n_1}[y^{m_1},x]x^{s_1}) \\ &= x^{n_1+n_2}[y^{m_1m_2},x]x^{s_1+s_2}. \end{aligned}$$

Let t be any positive integer. By repeated use of the above process, we obtain

$$x^{r_1+r_2+\cdots+r_t}[x,y] = x^{n_1+n_2+\cdots+n_t}[x,y^{m_1m_2\cdots m_t}]x^{s_1+s_2+\cdots+s_t} \text{ for all } x,y \in R, \tag{4}$$

or

$$x^{r_1+r_2+\cdots+r_t}[x,y] = x^{n_1+n_2+\cdots+n_t}[y^{m_1m_2\cdots m_t},x]x^{s_1+s_2+\cdots+s_t} \text{ for all } x,y \in R. \tag{4$'$}$$

If $u \in N(R)$, then by (4) and (4)$'$, for any $x \in R$ and any positive integer t, we have

$$x^{r_1+r_2+\cdots+r_t}[x,u] = x^{n_1+n_2+\cdots+n_t}[x,u^{m_1m_2\cdots m_t}]x^{s_1+s_2+\cdots+s_t},$$

or

$$x^{r_1+r_2+\cdots+r_t}[x,u] = x^{n_1+n_2+\cdots+n_t}[u^{m_1m_2\cdots m_t}, x]x^{s_1+s_2+\cdots+s_t}.$$

But $u^{m_1m_2\cdots m_t} = 0$ for sufficiently large t. Therefore, $x^{r_1+r_2+\cdots+r_t}[x,u] = 0$ and by Lemma 2, $[x,u] = 0$. Hence $N(R) \subseteq Z(R)$. So

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{5}$$

Remark 1. Since we know that $C(R) \subseteq Z(R)$, if R has a unity 1. Thus $[x,[x,y] = 0$ for all $x, y \in R$, and hence we shall apply the conclusion of Lemma 1 without explicit mention for any ring R satisfying the hypothesis of Lemma 8.

Lemma 9. *Let $m = m(y) > 1$ be an integer depending on y and n, r and s be fixed non-negative integers. If R is a ring with unity 1 satisfies $[x^r y \pm x^n y^m x^s, x] = 0 = [x^r y^m \pm x^n y^{m^2} x^s, x]$ for all $x, y \in R$, then R is commutative.*

Proof. If $r = n+s$, then $x^r[x,y] = \pm x^{n+s}[x,y^m] = \pm x^r[x,y^m]$. Thus $x^r([x,y] \mp [x,y^m]) = 0$ and by Lemma 2 $[x, y \mp y^m] = 0$. Therefore, R is commutative by Theorem H.

Let $r > n+s$. Suppose that $q_1 = p^{r+1} - p^{n+s+1}$ for a prime p. Then by (3) we have

$$\begin{aligned} q_1 x^r[x,y] &= p^{r+1}x^r[x,y] - p^{n+s+1}x^r[x,y] \\ &= (px)^r[(px),y] \mp (px)^n[(px),y^m](px)^s = 0. \end{aligned}$$

Similarly, if $n+s > r$, then for $q_2 = p^{n+s+1} - p^{r+1}$, we get $q_2 x^r[x,y] = 0$. Suppose $q = q_1$ or q_2. Then $q[x,y] = 0$ by Lemma 2. So $[x,y^q] = qy^{q-1}[x,y] = 0$ for all $x, y \in R$. Therefore,

$$y^q \in Z(R) \text{ for all } y \in R. \tag{6}$$

Further using (1) and (1′), together with Lemma 1, and Lemma 8 several times, we see that

$$\begin{aligned}(1-y^{(m-1)^2})[x,y^m]x^r &= x^r[x,y^m] - y^{(m-1)^2}[x,y^m]x^r \\ &= x^r[x,y^m] - my^{m-1}y^{(m-1)^2}[x,y]x^r \\ &= x^r[x,y^m] - my^{m(m-1)}x^r[x,y] \\ &= x^r[x,y^m] \mp my^{m(m-1)}x^n[x,y^m]x^s \\ &= x^r[x,y^m] \mp [x,y^{m^2}]x^{n+s} \\ &= x^r[x,y^m] \mp x^n[x,y^{m^2}]x^s \\ &= 0.\end{aligned}$$

This implies that $(1-y^{(m-1)^2})[x,y^m]x^r = 0$, that is, $(1-y^{(m-1)^2})[x,y^m]$ $x^{r+n+s} = 0$ So $(1-y^{(m-1)^2})[x,y]x^{2r} = 0$. Using Lemma 2, we get $(1-y^{(m-1)^2})$ $[x,y] = 0$. But since $y^q \in Z(R)$, for all $y \in R$, that gives $[x, y - y^{q(m-1)^2+1}] = (1-y^{q(m-1)^2})[x,y] = 0$ and therefore, R is commutative by Theorem H.

Proof of Theorem 1. If R is left s-unital satisfies (1), then R is s-unital by Lemma 7. In view of Proposition 1 of [12], we may assume R has unity 1. Therefore, R is commutative by Lemma 9.

Corollary 1. *Let $m > 1$, n, r and s be fixed non-negative integers. If R is a left s-unital ring satisfies $[x^r y \pm x^n y^m x^s, x] = 0$ for all $x, y \in R$, then R is commutative.*

In preparation for proving Theorem 2, we prove the following lemmas:

Lemma 10. *Let $m = m(x,y) > 1$, $n = n(x,y)$, $r = r(x,y)$ and $s = s(x,y)$ be non-negative integers, and let R be a right s-unital ring. If R satisfies $[yx^r \pm x^n y^m y^s, x] = 0$ for all $x, y \in R$, then R is s-unital.*

Proof. If $x, y \in R$, then there exists $e = e(x,y) \in R$ such that $xe = x$ and $ye = y$. Further, there exist non-negative integers $m = m(x,e) > 1$, $n = n(x,e) > 0$, $r = r(x,e)$, and $s = s(x,e)$ such that $x^{r+1} = ex^{r+1}$. Also, if $m_1 = m(y,e) > 1$, $n_1 = n(y,e) > 0$, $r_1 = r(y,e)$ and $s_1 = s(y,e)$, then we get $y^{r_1+1} = ey^{r_1+1}$. Thus

$ex^{r+r_1+2} = x^{r+r_1+2}$ and $ey^{r+r_1+2} = y^{r+r_1+2}$. Therefore, R is s-unital by Lemma 6. If $n = n(x,y) = 0$, then $y = (e \mp (ey^{m-1} - y^{m-1}))y \in Ry$, for $m = m(e,y) > 1$. Thus R is s-unital.

Lemma 11. *Let $m = m(x,y) > 1$, $n = n(x,y)$, $r = r(x,y)$ and $s = s(x,y)$ be non-negative integers. If R satisfies $[yx^r \pm x^n y^m x^s, x] = 0$ for all $x, y \in R$, then $C(R) \subseteq N(R)$. Further, if R has unity 1, then $C(R) \subseteq Z(R)$.*

Proof. By our hypothesis, we obtain

$$[x,y]x^r = \pm x^n [x, y^m] x^s \quad \text{for all } x, y \in R. \tag{11}$$

If $x = e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $y = e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ in $(GF(p))_2$ for any prime p, then x and y fail to satisfy (11). Therefore, $C(R) \subseteq N(R)$ by Lemma 4.

Following the proof of Lemma 8, we notice that for any positive integer k, (11) implies that

$$[x,y]x^{r_1+r_2+\cdots+r_k} = x^{n_1+n_2+\cdots+n_k}[x, y^{m_1 m_2 \cdots m_k}]x^{s_1+s_2+\cdots+s_k} \text{ for all } x, y \in R, \tag{12}$$

or

$$[x,y]x^{r_1+r_2+\cdots+r_k} = x^{n_1+n_2+\cdots+n_k}[y^{m_1 m_2 \cdots m_k}, x]x^{s_1+s_2+\cdots+s_k} \text{ for all } x, y \in R. \tag{12$'$}$$

Also we can prove that $N(R) \subseteq Z(R)$. Therefore,

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{13}$$

Proof of Theorem 2. In view of Lemma 10, R is s-unital. Hence we can assume that R has unity 1 as suggested by Proposition 1 of [12]. By (13), (2) and (2$'$) are equivalent to

$$x^r[x,y] = \pm x^n [x, y^m] x^s \quad \text{for all } x, y \in R,$$

$$x^r[x,y^m] = \pm x^n [x, y^{m^2}] x^s \quad \text{for all } x, y \in R.$$

Therefore, R is commutative by Lemma 9.

Corollary 2. *Let $m > 1$, n, r and s be fixed non-negative integers. If R is a right s-unital ring which satisfies the polynomial identity $[yx^r \pm x^n y^m x^s, x] = 0$ for all $x, y \in R$, then R is commutative.*

Remark 2. Let $r = n = 0$ (resp. $r = s = 0$) in (1) (resp. (2)). Then

$$[x, y] = \pm[y^m, x]x^s \ \textit{for all } x, y \in R \tag{14}$$

(resp.

$$[x, y] = \pm x^n[y^m, x] \ \textit{for all } x, y \in R \quad (15)).$$

If $m > 1$ or $s \geq 1$ in (14) (resp. $m > 1$ or $n \geq 1$ in (15)), then R is a $(\mathrm{Z}, \bar{\beta})$-ring in the sense of Streb ([19]), hence R is commutative even if R is not assumed to be a left (resp. right) s-unital ring (ring with unity 1 (cf. Lemma 9)).

Example 1. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of $(GF(2))_2$. It is easy to check that R is a right s-unital ring satisfying the polynomial identity $[x^r y \pm x^n y^m x^s, x] = 0$ for each $x, y \in R$, where $r > 1$, $n > 1$, $m = m(y) > 1$, and $s > 1$ are integers. Also, R is not a left s-unital ring. However, R is a non-commutative ring.

Example 2. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & o \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of $(GF(2))_2$. It is easy to check that R is a left s-unital ring satisfying the polynomial identity $[yx^r \pm x^n y^m x^s, x] = 0$ for all $x, y \in R$, where $r > 1$, $n > 1$, $m = m(y) > 1$, and $t > 1$ are integers. Also, R is not a right s-unital ring. However, R is a non-commutative ring.

Lemma 12. *Let $m = m(x, y) > 1$, $r = r(x, y)$ and $s = (x, y)$ be non-negative integers. If R is a right s-unital ring satisfies $[x^r y \pm y^m x^s, x]$ for all $x, y \in R$, then R is s-unital.*

Proof. Since R is right s-unital, then for any $x, y \in R$ there exists an element $e = e(x,y) \in R$ such that $xe = x$ and $ye = y$. Let $m = m(x,e) > 1$, $r = r(x,e) \geq 1$, $s = s(x,e) \geq 0$, $m' = m(y,e) > 1$, $r' = r(y,e) \geq 1$, and $s' = s(y,e) \geq 0$. Then $e^{mm'}x^{s+s'+2} = x^{s+s'+2}$, and $e^{mm'}y^{s+s'+2} = y^{s+s'+2}$. By Lemma 6, R is s-unital. If $r = r(x,e) = 0$, then $[e,y] = [e,y^m]e^s$ for $s = s(e,y) \geq 0$, and $m = m(e,y) > 1$. So $y = (e \mp (ey^{m-1} - y^{m-1}))y \in Ry$. Therefore, R is an s-unital ring.

Theorem 3. *Let $m = m(y) > 1$ and r, s be non-negative integers. If R is a right s-unital ring satisfies $[x^r y \pm y^m x^s, x] = 0 = [x^r y^m \pm y^{m^2} x^s, x]$ for all $x, y \in R$, then R is commutative.*

Proof of Theorem 3. By Lemma 12, R is an s-unital ring. Hence, we can assume that R has unity 1 (see [12, Proposition 1]). Therefore, R is commutative by Lemma 9.

Lemma 13. *Let $m = m(x,y) > 1$, $r = r(x,y)$ and $n = (x,y)$ be non-negative integers. If R is a left s-unital ring satisfies $[yx^r \pm x^n y^m, x]$ for all $x, y \in R$, then R is s-unital.*

Proof. Let R be a left s-unital ring. Then for any $x, y \in R$ there exists an element $e = e(x,y) \in R$ such that $ex = x$ and $ey = y$. Let $m = m(x,e) > 1$, $r = r(x,e) \geq 1$, $n = n(x,e) \geq 0$, $m' = m(y,e) > 1$, $r' = r(y,e) \geq 1$, and $n' = n(y,e) \geq 0$. Then $x^{n+n'+2} = x^{n+n'+2}e^{mm'}$, and $y^{n+n'+2} = y^{n+n'+2}e^{mm'}$. By Lemma 6, R is s-unital. If $r = r(r,e) = 0$, then $[e,y] = \pm e^n[e,y^m]$, for $n = n(e,y) \geq 0$, and $m = m(e,y) > 1$. Thus $y = y(e \pm (y^{m-1} - y^{m-1}e)) \in yR$. Therefore, R is an s-unital ring.

Theorem 4. *Let $m = m(y) > 1$ and r, s be non-negative integers. If R is a left s-unital ring satisfies $[yx^r \pm x^n y^m, x] = 0 = [y^m x^r \pm x^n y^{m^2}, x]$ for all $x, y \in R$, then R is commutative.*

Proof of Theorem 4. By Lemma 13, R is an s-unital ring. Hence, we can assume that R has unity 1 by Proposition 1 of [12]. Therefore, R is commutative by Lemma 9.

Corollary 3 ([10, Theorem 4]. *Let R be a ring with unity 1, and let $n \geq 1$ be a fixed integer, and suppose that for each $y \in R$, there exists an integer $m = m(y) > 1$ such that $[x, xy - x^n y^m] = 0$ for all $x \in R$. Then R is commutative.*

Remark 3. The example of Grassman algebra rules out the possibility that $m = 1$ in Lemma 9 and therefore, Theorems 1-4.

Theorem 5. *Let r be a fixed non-negative integer. If R is a left (resp. right) s-unital ring satisfies*

$$x^r[x, y] = 0 \; for \; all \; x, y \in R \tag{16}$$

(resp.

$$[x, y]x^r = 0 \; for \; all \; x, y \in R, \qquad (17))$$

then R is commutative.

Proof. Let $x, y \in R$. Then there exists $e = e(x, y) \in R$ (resp. $f = f(x, y) \in R$) such that $ex = x$ and $ey = y$ (resp. $xf = x$ and $yf = y$). Thus $y = ye$ (resp. $y = fy$). Similarly, $x = xe$ (resp. $x = fx$). Therefore, R is s-unital. By Proposition 1 of [12], we may assume that R has unity 1. Then $x^r[x, y] = 0 = (x+1)^r[x, y$ (resp. $[x, y]x^r = 0 = (x+1)^n[x, y]$) for all $x, y \in R$. By Lemma 2, $[x, y] = 0$ and thus R is commutative.

Remark 4. In case $r > 0$ Theorem 3 need not be true for right (resp. left) s-unital ring. Indeed, we have the following:

Example 3. Let K be any field. Then the non-commutative ring $R = \begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ (resp. $R^* = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$) has a right (resp. left) identity element and satisfies $x[x, y] = 0$ (resp. $[x, y]x = 0$) for all $x, y \in R$. Also R is not s-unital ring.

Example 4. If we drop the restriction that R with unity 1 in Lemma 9, then the ring R may be badly non-commutative. Indeed, we let D_k be the ring of all $k \times k$ matrices over a division ring D, and let

$$A_k = \{ (a_{ij}) \in D_k \mid a_{ij} = 0, \ j \geq i \}.$$

Then A_k is a non-commutative nilpotent ring of index k, for any positive integer $k > 2$. Clearly, A_3 satisfies (1) and (2).

Example 5. Let F be a field. Define an algebra $R = A$ over F with a basis $\{f_1, f_2, f_3\}$ where $f_1 f_2 = f_2$ and all other products are zero. Then A is nilpotent of index 3 satisfies the identities (1) and (2). R is not commutative.

In the remaining case we suppose that $m = 1$ in (1) and (2).

Theorem 6. *Let n, r and s be fixed non-negative integers, and let R be an s-unital ring satisfying*

$$x^r[x,y] = \pm x^n[x,y]x^s \ \textit{for all } x, y \in R. \tag{18}$$

Then R is commutative in any of the following:

(i) R satisfies $[x,y] = -[x,y]$, and R is 2-torsion free.

(ii) $0 = s = n < r$.

(iii) $0 < s < r$, $n = 0$ and R is $r!$-torsion free.

(iv) $0 < n < r$, $s = 0$ and R is $r!$-torsion free.

(v) $r = 0$ and $n > 0$ or $s > 0$.

Proof. According to [12, Proposition 1], we may assume that R has unity 1.

(i) By hypothesis, $2[x,y] = 0$. Therefore, R is commutative, since R is 2-torsion free.

(ii) The identity (18) becomes $x^r[x,y] = \pm[x,y]$ for all $x, y \in R$. Therefore R is commutative by [6, Theorem].

(iii) Let $I_0^r(x) = x^r$ and $I_0^s(x) = x^s$. Then the polynomial identity (18) gives

$$x^r[x,y] = \pm[x,y]x^s,$$

and hence

$$I_0^r(x)[x,y] = \pm[x,y]I_0^s(x) \quad \text{for all } x,y \in R.$$

Replace x by $x+1$ in the last identity to get $I_0^r(x+1)[x,y] = \pm[x,y]I_0^s(x+1)$. By Lemma 5, we have $I_1^r(x)[x,y] = \pm[y,x]I_1^s(x)$. Again, replace x by $x+1$ and apply Lemma 5 to obtain $I_2^r(x)[x,y] = \pm[x,y]I_2^s(x)$. Now iterating the last identity r times, we finally get

$$I_r^r(x)[x,y] = \pm[x,y]I_r^s(x) \quad \text{for all } x,y \in R. \tag{19}$$

Since by Lemma 5, $I_r^r(x) = r!$ and $I_r^s(x) = 0$ for $r > s$, the identity (19) reduces to $r![x,y] = 0$. As every commutator in R is $r!$-torsion free, we get $[x,y] = 0$ for all $x,y \in R$. Therefore R is commutative.

(iv) Similar to the proof of case (iii).

(v) Without loss of generality suppose that $n > 0$. Then we have

$$[x,y] = \pm x^n[x,y]x^s \quad \text{for all } x,y \in R, \tag{20}$$

and thus, R is commutative by [19, Hauptsatz].

Remark 5. In Theorem 4 (i), (ii) and (v), R is not necessarily to be an s-unital ring (ring with unity 1).

Theorem 7. *Let r, n and s be fixed non-negative integers, and let R be an s-unital ring satisfying*

$$[x,y]x^r = \pm x^n[x,y]x^s \quad \textit{for all } x,y \in R. \tag{21}$$

Then R is commutative in any of the following:

(*i*) $0 = s = n < r$.

(*ii*) $0 < s < r$, $n = 0$ *and R is $r!$-torsion free.*

(*iii*) $0 < n < r$, $s = 0$ *and R is $r!$-torsion free.*

(*iv*) $r = 0$ *and* $n > 0$ *or* $s > 0$.

Theorem 8. *Let r, n and s be fixed non-negative integers such that $r \neq n+s$. Suppose that R is an s-unital ring satisfying the polynomial identity (18).*

Further, if every commutator in R *is* $| p^{r+1} - p^{n+s+1} |$*-torsion free for an integer* $p > 1$, *then* R *is commutative.*

Proof. According to Proposition 1 of [12], we can assume that R has unity 1. Thus

$$(px)^r[(px), y] = \pm(px)^n[(px), y](px)^s \text{ for all } x, y \in R. \tag{22}$$

By using (18) and (22), we obtain

$$| p^{r+1} - p^{n+s+1} | x^r[x, y] = 0 \text{ for all } x, y \in R. \tag{23}$$

By Lemma 2 and the hypothesis, (23) yields $[x, y] = 0$ for all $x, y \in R$. Therefore R is commutative.

Theorem 9. *Let* r, n *and* s *be fixed non-negative integers such that* $r \neq n + s$. *Suppose that* R *is an s-unital ring satisfying the polynomial identity (21). Further, if every commutator in* R *is* $| p^{r+1} - p^{n+s+1} |$*-torsion free for an integer* $p > 1$, *then* R *is commutative.*

Next, we suppose that $r > 0$, $n > 0$ and $s > 0$ in (18) and (21). Indeed we prove the following:

Theorem 10. *Let* r, n *and* s *be fixed positive integers and let* R *be an s-unital ring satisfying the polynomial identity* (18). *If, further,* $N(R) \subseteq Z(R)$, *then* R *is commutative provided that* $r \neq n + s$ *and every commutator in* R *is* $r!$ *resp.* $(n + s)!$*-torsion free for* $r > n + s$, *resp.* $r < n + s$.

Proof. It is easy to see that $C(R) \subseteq Z(R)$. Thus $x^r[x, y] = \pm[x, y]x^{n+s}$ for all $x, y \in R$. Therefore, R is commutative by Theorem 4 (iii).

Theorem 11. *Let* r, n *and* s *be fixed positive integers and let* R *be an s-unital ring satisfying the polynomial identity (21). If, further,* $N(R) \subseteq Z(R)$, *then* R *is commutative provided that* $r \neq n + s$ *and every commutator in* R *is* $r!$ *(resp.* $(n + s)!$*)-torsion free for* $r > n + s$ *(resp.* $r < n + s$*).*

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