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APPLICATIONS OF THE KKM-PRINCIPLE TO PROLLA TYPE THEOREMS

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Abstract. We prove some theroems of Prolla type [13] using a well known KKM-principle of Ky Fan [6], so generalizing several results known in the literature.

The following theorem due to Prolla [13] was proved using tools from approximation theory and the Bohnenblust and Karlin theorem [2]:

Theorem 1. *Let X be a nonempty compact convex subset of a normed linear space E and $g : X \to X$ be a continuous almost affine onto map. Then for each continuous map $f : X \to E$, there exists a point $x_0 \in X$ such that*

$$\|gx_0 - fx_0\| = \inf\{\|x - fx_0\| : x \in X\}. \tag{1}$$

Let X be nonempty convex subset of a normed linear space E and $g : X \to E$. We recall that g is almost affine on X if

$$\|g(\lambda x_1) + (1-\lambda)x_2) - y\| \leq \lambda \cdot \|gx_1 - y\| + (1-\lambda) \cdot \|gx_2 - y\|$$

for all $x_1, x_2 \in X$, $\lambda \in [0,1]$ and $y \in E$.

The following result, due to Ky Fan [6], extends known results on the KKM-principle.

Theorem 2. *Let Y be a nonempty convex subset of a Hausdorff topological vector space E and X be a nonempty subset of Y. For each $x \in X$, let Fx be a*

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relatively closed subset of Y such that F is a KKM-map. If there is a nonempty subset X_0 of X such that the intersection $B = \cap_{x \in X_0} Fx$ is compact and X_0 is contained in a compact convex subset S of Y, then $\cap_{x \in X} Fx \neq \phi$.

Remark 1. As noted by Lin [11], the set B is necessarily nonempty.

A very good source of reference on KKM-Principle is due to Granas [7] where several applications are given. The following result due to Allen [1] follows from Theorem 2 as a Corollary.

Theorem 3. *Let X be a nonempty convex subset of a Hausdorff topological vector space. Let $\Phi : X \times X \to R$ be a real valued function satisfying the following properties:*

(i) *For each fixed $x \in X$, $\Phi(x,y)$ is a lower semicontinuous function of y on X,*

(ii) *For each fixed $y \in X$, $\Phi(x,y)$ is a quasiconcave function of x on X,*

(iii) *$\Phi(x,x) \leq 0$ for all $x \in X$,*

(iv) *X has a nonempty compact convex subset X_0 such that the set $B = \{y \in X : \Phi(x,y) \leq 0$ for all $x \in X_0\}$ is compact.*

Then there exists some $y_0 \in X$ such that $\Phi(x, y_0) \leq 0$ for all $x \in X$.

We recall that a real function f on a convex set X is quasiconcave if the $\{x \in X : f(x) > t\}$ is convex for all $t \in R$ and it is lower semicontinuous if the set $\{x \in X : f(x) \leq t\}$ is closed in X for all $t \in R$.

Remark 2. As noted by Shih and Tan [14], the coercive condition (iv) is a unification of the two coercive conditions given by Allen [1] and Brezis, Nirenberg and Stampacchia [3]. It is easily seen that (ii) and (iii) imply that B is nonempty.

We prove the following:

Theorem 4. *Let X be a nonempty compact convex of a normed linear space E and let $g : X \to E$ be a continuous map such that*

(*a*) *$g(X)$ is convex and $g^{-1}(y)$ is convex for every $y \in g(X)$.*

Then for each continuous map $f : X \to E$, either there exists some $x_0 \in X$ such that $gx_0 = fx_0$ or for any $y \in g(X)$:

$$0 < \|gx_0 - fx_0\| \leq \|y - fx_0\|.$$

Remark 3. We note that the almost affine map $g : X \to X$ of Theorem 1 satisfies condition (a). Indeed, let $y \in g(X) = X$ and $x_1, x_2 \in g^{-1}(y)$. Then we have for each $\lambda \in [0,1]$:

$$\|g(\lambda x_1 + (1-\lambda)x_2) - y\| \leq \lambda \cdot \|gx_1 - y\| + (1-\lambda) \cdot \|gx_2 - y\| = 0,$$

i.e. $\lambda x_1 + (1-\lambda)x_2 \in g^{-1}(y)$. Hence $g^{-1}(y)$ is convex for any $y \in g(X)$. This implies that Theorem 1 follows from Theorem 4.

We need the following theorem, which was established by Komiya [10] combining Lemma 1 of Ha [8] and Prop. 2 of Browder [4]:

Theorem 5. *Let A be a nonempty convex subset of a Hausdorff linear topological space E and let B be a nonempty compact convex subset of a Hausdorff linear topological space F. Let $S : A \to 2^B$ be an upper semicontinuous set-valued mapping such that $S(a)$ is a nonempty closed convex subset of B for each $a \in A$ and $T : B \to 2^A$ be a set-valued mapping such that $T(b)$ is a nonempty convex subset of A and $T^{-1}(a) = \{b \in B : a \in T(b)\}$ is open in B for each $a \in A$. Then there exist a point $a_0 \in A$ and a point $b_0 \in B$ such that $a_0 \in T(b_0)$ and $b_0 \in S(a_0)$.*

We recall that a set-valued mapping $S : A \to 2^B$ is upper semicontinuous if $S^{-1}(C) = \{a \in A : S(a) \cap C \neq \phi\}$ is closed in A for every closed subset C of B.

Proof of theorem 4. Assuming the negation of the second alternative, we have (b) for each $x \in X$ such that $\|gx - fx\| > 0$, there exists a point $y \in g(X)$ such that

$$\|gx - fx\| > \|y - fx\|.$$

Let $gx \neq fx$ for any $x \in X$. Define a set-valued mapping $T : X \to 2^{g(X)}$ by setting $T(x) = \{y \in g(X) : \|gx - fx\| > \|y - fx\|\}$ for any $x \in X$.

Since $g(X)$ is convex and (b) holds, $T(x)$ is a nonempty convex subset of $g(X)$. Further, $X\backslash T^{-1}(y) = \{x \in X : \|gx - fx\| \leq \|y - fx\|\}$ is closed in X since f and g are continuous, so $T^{-1}(y)$ is open in X for any $y \in g(X)$. Now, define a set-valued mapping $S : g(X) \to 2^X$ by setting $S(y) = g^{-1}(y)$ for each $y \in g(X)$. Since g is closed and $S^{-1}(C) = g(C)$ for any closed subset C of X, we have that S is an upper semicontinuous mapping such that, by (a), $S(y)$ is a nonempty closed convex subset of X for each $y \in g(X)$. Using Theorem 5 with $A = g(X)$ and $B = X$, there exists two points $y_0 \in g(X)$ and $x_0 \in X$ such that $y_0 \in Tx_0$ and $x_0 \in g^{-1}(y_0)$. This implies that $\|gx_0 - fx_0\| > \|y_0 - fx_0\| = \|gx_0 - fx_0\|$, a contradiction.

Remark 4. It is clear that the above proof can be adopted to prove the more general Theorem 3 of Ha [9], of which Theorem 4 is a special case.

Now we prove some results where the compactness on the referential set X is relaxed. In the sequel, w stands for the weak topology.

Theorem 6. *Let X be a nonempty convex subset of a normed linear space E. Let $f : (X, w) \to (E, \|\ \|)$ be a sequentially strongly continuous map and $g : (X, w) \to (E, w)$ be a sequentially weakly continuous map such that*

(a_1) *$g^{-1}([y, z])$ is convex for $y, z \in g(X)$.*

(b_1) *Moreover, let X_0 be a nonempty weakly compact convex subset of X such that the set $B = \{y \in X : \|gy - fy\| \leq \|gx - fy\|$ for all $x \in X_0\}$ is weakly compact. Then there exists a point $x_0 \in B$ such that*

$$\|gx_0 - fx_0\| \ = \ \inf\{\|gx - fx_0\| : x \in X\}. \tag{2}$$

If $g(X) = X$, then (1) is satisfied.

Proof. Define a set-valued mapping $F : X \to 2^X$ by setting $F(x) = \{y \in X : \|gy - fy\| \leq \|gx - fy\|\}$ for each $x \in X$. Fx is weakly closed. Indeed, let $\{y_\alpha\}$ be in Fx converging weakly to y. Then $gy_\alpha - fy_\alpha \to gy - fy$ weakly and $gx - fy_\alpha \to gx - fy$ strongly. Now,

$$\|gy - fy\| \leq \liminf_{\alpha} \|gy_\alpha - fy_\alpha\| \leq \liminf_{\alpha} \|gx - fy_\alpha\| = \|gx - fy\|,$$

i.e. $y \in Fx$. We show that F is a KKM-map, i.e. the convex hull co $\{x_1, x_2, \dots, x_n\}$ of every finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} Fx_i$. Indeed, let $z \in co\{x_1, \dots, x_n\}$ and assume that $z \notin Fx_i$ for any $i = 1, \dots, n$. Then $x_i \notin F^{-1}(z)$, i.e. $x_i \in X \setminus F^{-1}(z) = \{x \in X : \|gz - fz\| > \|gx - fz\|\}$ for any $i = 1, \dots, n$. As proved in Theorem 1 of Lin [12], $X \setminus F^{-1}(z)$ is convex. Indeed, let $z_1, z_2 \in X \setminus F^{-1}(z)$ and let $gz_1 = u_1$, $gz_2 = u_2$. Since $g^{-1}([u_1, u_2])$ is convex by (a_1), we have $\lambda z_1 + (1-\lambda)z_2 \in g^{-1}([u_1, u_2])$ for $\lambda \in [0,1]$. Thus $g(\lambda z_1 + (1-\lambda)z_2) \in [u_1, u_2]$, i.e. for any $\lambda \in [0,1]$ there exists $h_\lambda \in [0,1]$ such that $g(\lambda z_1 + (1-\lambda)z_2) = h_\lambda u_1 + (1 - h\lambda)u_2 = h_\lambda g(z_1) + (1 - h_\lambda)g(z_2)$.

Then

$$\|g(\lambda z_1 + (1-\lambda)z_2) - fz\| = h_\lambda \cdot \|gz_1 - fz\| + (1 - h_\lambda) \cdot \|gz_2 - fz\| < \|gz - fz\|.$$

This means that $\lambda z_1 + (1-\lambda)z_2 \in X \setminus F^{-1}(z)$ for $\lambda \in [0,1]$, i.e. $X \setminus F^{-1}(z)$ is convex. Then co $\{x_1, \dots, x_n\} \subset X \setminus F^{-1}(z)$, i.e. $z \notin F^{-1}(z)$, a contradiction to the fact $x \in F(x)$, i.e. $x \in F^{-1}(x)$ for each $x \in X$. Since (b_1) holds, all the hypothesis of Theorem 2 (with $X = Y$ and $S = X_0$) are satisfied in (E, w). Then $\cap_{x \in X} F(x) \neq \phi$ and a point x_0 of this intersection, contained in B, verifies our conclusion.

Remark 5. Condition (a_1) is the same condition (c') used by Ha [9] and Lin [12].

Corollary 1. *Let X be a nonempty weakly compact convex subset of a normed linear space E. Let f, g satisfy conditions of Theorem 6 including* (a_1). *Then there exists a point $x_0 \in X$ satisfying* (2).

Proof. It suffices to observe that condition (b_1) of Theorem 6 is fulfilled in this case by taking any nonempty weakly closed convex subset X_0 (in particular, X itself) of X.

Corollary 2. *Let X be a nonempty compact convex subset of a normed linear space E, let $f, g : X \to E$ be continuous and g satisfies condition* (a_1). *Then there exists a point $x_0 \in X$ such that* (2) *holds.*

Proof. In this case, $(X, w) \equiv (X, \| \ \|)$ since $(X, \| \ \|)$ is strongly compact and (X, w) is separated. Consequently, the concepts of continuity and sequential continuity coincide since X is metrizable. Thus the thesis follows from Corollary 1.

Remark 6. Corollary 2 is also a corollary of Theorem 3 of Ha [9] (with condition (a_1)) by taking $E = F$ as normed linear space. Corollary 2 is also a corollary of Theorem 2 of Lin [12] by assuming $E = F$. Note that Lin [12] derived his Theorem 1 (which is Theorem 3 of Ha [9] under condition (a_1)) directly by the famous Lemma 1 of Fan [5] and his Theorem 2 from his Lemma [11], which includes Lemma 1 of Fan [5]. Following the lines of proof of Theorem 6, it is easy to derive Theorem 2 of Lin [12] directly from Theorem 2.

Remark 7. As pointed out by Lin [11], [12], condition (b_1) of Theorem 6 can be replaced by the following condition [6, Theorem 7], [15]:

(b_1') Let X_0 be a nonempty weakly compact convex subset of X and K be a nonempty weakly compact subset of X such that for every $y \in X \setminus K$, there exists a point $x \in X_0$ for which $\|gy - fy\| > \|gx - fy\|$. The conclusion of Theorem 6 will be: there exists a point $x_0 \in K$ such that (2) holds.

We state and prove the following using Theorem 3.

Theorem 7. *Let X be a nonempty convex subset of a normed linear space E, $f : (X, w) \to (E, \| \ \|)$ be sequentially strongly continuous and $g : (X, w) \to (E, w)$ be sequentially weakly continuous and almost affine on X. Moreover, condition (b_1) holds. Then there exists a point $x_0 \in B$ such that (2) holds. If $g(X) = X$, then (1) is satisfied.*

Proof. Define $\Phi : X \times X \to R$ by $\Phi(x, y) = \|gy - fy\| - \|gx - fy\|$ for all $x, y \in X$. For each $x \in X$, then $\Phi(x, y)$ is a weakly lower semicontinuous function of y on X(cfr. proof of Theorem 6). For any $y \in X$ and $t \in R$, let $C_t(y) = \{x \in X : \Phi(x, y) > t\}$. We show that this set is convex. If $x_1, x_2 \in C_t(y)$

and $\lambda \in [0,1]$, we have

$$\begin{aligned} &\Phi(\lambda x_1 + (1-\lambda)x_2, y) \\ &= \|gy - fy\| - \|g(\lambda x_1 + (1-\lambda)x_2 - fy\| \\ &\geq \|gy - fy\| - \lambda \cdot \|gx_1 - fy\| - (1-\lambda) \cdot \|gx_2 - fy\| \\ &> \|gy - fy\| + \lambda(t - \|gy - fy\|) + (1-\lambda)(t - \|gy - fy\|) \\ &= t, \end{aligned}$$

since g is almost affine. All the conditions of Theorem 3 are satisfied and the thesis follows.

Remark 8. Of course, condition (b_1) can be replaced in Theorem 7 by condition (b_1').

Corollary 3. *Let X be a nonempty weakly compact convex of a normed linear space E. Let f, g be as in Theorem 7 and $g(X) = X$. Then there exists a point $x_0 \in X$ satisfying* (1).

Remark 9. Theorem 1 is clearly a consequence of Corollary 3. If g is the identity function of X, Theorem 6 or Theorem 7 give Theorem 3 of Singh, Sehgal and Smithson [15].

Remark 10. It is evident that Theorem 6 and Theorem 7 can be established in the more general context of a locally convex Hausdorff topological vector space E. In this case, as observed by Lin [12], conditions (b_1) and (b_1') are replaced respectively by the following:

(c_1) For any continuous seminorm p on E, there exists a nonempty weakly compact convex subset $X_0(p)$ of X such that the set $B(p) = \{y \in X : p(gy - fy) \leq p(gx - fy)$ for all $x \in X_0(p)\}$ is weakly compact.

(c_1') For any continuous seminorm p on E, there exists a nonempty weakly compact convex subset $X_0(p)$ of X and a nonempty weakly compact subset $K(p)$ such that for every $y \in X \setminus K(p)$, there exists a point $x \in X_0(p)$ for which $p(gy - fy) \leq p(gx - fy)$.

Of course, in Theorem 7 one defines that g is almost affine on X for any continuous seminorm p, $\lambda \in [0,1]$, $x_1, x_2 \in X$, $y \in E$, it is $p(g(\lambda x_1 + (1-\lambda)x_2) - y) \leq \lambda p(gx_1 - y) + (1-\lambda)p(gx_2 - y)$.

In this case, the proof of Theorem 6 and 7 is deduced via the Hahn-Banach theorem (cfr. proof of Lemma 1 of [15]) and the conclusion will be:

Either there exists a point $x_0 \in X$ such that $gx_0 = fx_0$ or there exists a continuous seminorm p on F and a point $x_0 \in B(p)$ (resp. $x_0 \in K(p)$), if (c_1) (resp. (c_1')) is assumed, such that $0 < p(gx_0 - fx_0) \leq p(y - fx_0)$ for all $y \in g(X)$.

In this way, of course, for $g =$ identity function on X, one deduces Theorem 1 of [15].

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