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A NOTE ON TWO SUMMABILITY METHODS

W. T. SULAIMAN

Abstract. A theorem connecting the two summabilities $|\overline{N},p_n|_k$ and $|C,\alpha|_k$ is proved. This theorem contains as special cases the result of Mohapatra 1967.

1. Introduction

Let Σa_n be an infinite series of partial sums s_n. Let σ_n^δ and η_n^δ denote the nth Cesaro mean of order $\delta(\delta > -1)$ of the sequences $\{s_n\}$ and $\{na_n\}$ respectively. The series Σa_n is said to be absolutely summable (C,δ) with index k, or simply summable $|C,\delta|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|\sigma_n^\delta - \sigma_{n-1}^\delta|^k < \infty,$$

or equivalently

$$\sum_{n=1}^{\infty} \frac{1}{n}|\eta_n|^k < \infty.$$

Let $\{p_n\}$ be a sequence of positive real constants such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty \quad (P_{-1} = p_{-1} = 0).$$

A series Σa_n is said to be summable $|\overline{N},p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|T_n - T_{n-1}|^k < \infty \quad \text{(Bor 1986)}$$

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where

$$T_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v.$$

For $p_n = \frac{1}{n+1}$, the summability $|\overline{N}, p_n|$, which is the same as $|\overline{N}, p_n|_1$, is equivalent to the summability $|R, \log n, 1|$, and for $p_n = 1$, $|\overline{N}, p_n|_k$ summability is equivalent to $|C, 1|_k$ summability. In general the two summability methods $|C, \delta|_k$ and $|\overline{N}, p_n|_k$ are independent of each other.

The following results are known:

Theorem A (Mohapatra 1967) *Let sequence $\{\in_n\}$ satisfy the following*

$$\in_n = o(1) \quad as \quad n \to \infty;$$

$$\sum_{n=1}^{\infty} n^{-\alpha+1} \lambda_n \log(n+1) |\Delta \in_n| < \infty \quad (0 < \alpha \leq 1);$$

$$\sum_{n=1}^{\infty} \lambda_n \log(n+1) |\Delta \in_n| < \infty \quad (\alpha > 1);$$

$$\sum_{n=1}^{\infty} \lambda_n \; n \log(n+1) |\Delta^2 \in_n| < \infty;$$

where $\{\lambda_n\}$ is a positive non-diminishing sequence of n. Then $\sum a_n \in_n$ is summable $|C, \alpha|$ $(\alpha > 0)$, whenever

$$\sum_{n=1}^{m} |T_n - T_{n-1}| = O \quad (\lambda_m).$$

We prove the following

Theorem 1. *Let $\{p_n\}$ be a sequence of positive numbers such that $np_n = O(P_n)$ and $\{P_n/np_n\}$ non-decreasing. Let $\{\in_n\}$ be such that $\in_n = o(1)$. Let $\{\lambda_n\}$ be positive non-decreasing such that*

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{k-1} |\Delta T_{n-1}|^k = O(\lambda_m^k), \quad m \to \infty$$

where T_n is the $(\overline{N}, p_n)$-mean of the $\sum a_n$. If

$$\sum_{n=1}^{\infty} n^{k-k\alpha-1} \frac{P_n}{p_n} |\Delta \in_n| \lambda_n^k < \infty \quad (0 < \alpha < 1)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{P_n}{p_n} |\Delta \in_n| \lambda_n^k < \infty \quad (\alpha \geq 1)$$

and

$$\sum_{n=1}^{\infty} \frac{P_n}{p_n} |\Delta^2 \in_n| \lambda_n^k < \infty,$$

then the series $\sum a_n \in_n$ is summable $|C, \alpha|_k$, $k \geq 1$, $\alpha > 0$.

Remark. If we put $p_n = \dfrac{1}{n+1}$, $k = 1$ in Theorem 1, we obtain Theorem A.

2. Lemma

We required the following Lemma for the proof of the Theorem Lemma [3] - If $\sigma > \delta > 0$, then

$$\sum_{n=v+1}^{m} \frac{(n-v)^{\delta-1}}{n^{\sigma}} = O(v^{\delta-\sigma}), \quad m \to \infty.$$

3. Proof of Theorem 1

Let t_n^{α} be the nth (C, α)-mean, $\alpha > 0$, of the sequence $\{na_n \in_n\}$. Then we have

$$t_n^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \in_v, \quad T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v.$$

$$\begin{aligned} t_n^{\alpha} &= \frac{1}{A_n^{\alpha}} \sum_{v=1}^{n} P_{v-1} a_v \{v A_{n-v}^{\alpha-1} P_{v-1}^{-1} \in_v\} \\ &= \frac{1}{A_n^{\alpha}} [\sum_{v=1}^{n-1} \{\sum_{r=1}^{v} P_{r-1} a_r\} \Delta_v \{v A_{n-v}^{\alpha-1} P_{v-1}^{-1} \in_v\} + \{\sum_{v=1}^{n} P_{v-1} a_v\} n P_{n-1}^{-1} \in_n] \end{aligned}$$

$$= \frac{1}{A_n^\alpha}[\sum_{v=1}^{n-1}\{vA_{n-v}^{\alpha-1} \in_v \Delta T_{v-1} + \frac{P_{v-1}}{P_v} A_{n-v}^{\alpha-1} \in_v \Delta T_{v-1} + (v+1)\frac{P_{v-1}}{P_v}$$

$$\Delta_v A_{n-v}^{\alpha-1} \in_v \Delta T_{v-1} + (v+1)\frac{P_{v-1}}{P_v} A_{n-v-1}^{\alpha-1} \Delta \in_v \Delta T_{v-1}\} + n\frac{P_n}{P_n} \in_n \Delta T_{n-1}]$$

$$= t_{n,1}^\alpha + t_{n,2}^\alpha + t_{n,3}^\alpha + t_{n,4}^\alpha + t_{n,5}^\alpha, \quad \text{say.}$$

In order to prove the Theorem, by Minkowski's inequality, it is therefore sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_{n,r}^\alpha|^k < \infty, \quad r = 1,2,3,4,5.$$

Applying Hölder's inequality,

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n} |t_{n,1}^\alpha|^k &\le \sum_{n=2}^{m+1} \frac{1}{nA_n^\alpha} \sum_{v=1}^{n-1} v^k A_{n-v}^{\alpha-1} | \in_v |^k |\Delta T_{v-1}|^k \{\sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^\alpha}\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^k | \in_v |^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{A_{n-v}^{\alpha-1}}{nA_n^\alpha} \\
&= O(1) \sum_{v=1}^{m} v^k | \in_v |^k |\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{\alpha-1}}{n^{1+\alpha}} \\
&= O(1) \sum_{v=1}^{m} v^{k-1} | \in_v | \, |\Delta T_{v-1}|^k \\
&= O(1) \sum_{v=1}^{\infty} (\frac{P_v}{p_v})^{k-1} |\Delta T_{v-1}|^k | \sum_{n=v}^{\infty} \Delta \in_n | \\
&= O(1) \sum_{v=1}^{\infty} (\frac{P_v}{p_v})^{k-1} |\Delta T_{v-1}|^k \sum_{n=v}^{\infty} |\Delta \in_n | \\
&= O(1) \sum_{n=1}^{\infty} |\Delta \in_n | \sum_{v=1}^{n} (\frac{P_v}{p_v})^{k-1} |\Delta T_{v-1}|^k \\
&= O(1) \sum_{n=1}^{\infty} |\Delta \in_n | \lambda_n^k \\
&= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{P_n}{p_n} |\Delta \in_n | \lambda_n^k
\end{aligned}$$

$$\sum_{n=2}^{m+1} \frac{1}{n}|t_{n,2}^{\alpha}|^k \leq \sum_{n=2}^{m+1} \frac{1}{nA_n^{\alpha}} \sum_{v=1}^{n-1} (\frac{P_v}{p_v})^k A_{n-v}^{\alpha-1} | \in_v |^k |\Delta T_{v-1}|^k \{\sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{A_n^{\alpha}}\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^k | \in_v |^k |\Delta T_{v-1}|^k \sum_{v=1}^{n-1} \frac{A_{n-v}^{\alpha-1}}{nA_n^{\alpha}}$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{v} (\frac{P_v}{p_v})^k | \in_v |^k |\Delta T_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{\infty} \frac{1}{v} (\frac{P_v}{p_v})^k |\Delta T_{v-1}|^k \sum_{n=v}^{\infty} |\Delta \in_n |$$

$$= O(1) \sum_{n=1}^{\infty} |\Delta \in_n | \sum_{v=1}^{n} \frac{P_v}{vp_v} (\frac{P_v}{p_v})^{k-1} |\Delta T_{v-1}|^k$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{P_n}{p_n} |\Delta \in_n | \sum_{v=1}^{n} (\frac{P_v}{p_v})^{k-1} |\Delta T_{v-1}|^k$$

$$= O(1) \sum_{n=1}^{\infty} \frac{1}{n} \frac{P_n}{p_n} |\Delta \in_n | \lambda_n^k$$

$$\sum_{n=2}^{m+1} \frac{1}{n}|t_{n,4}^{\alpha}|^k \leq \sum_{n=2}^{m+1} \frac{1}{n(A_n^{\alpha})^k} \sum_{v=1}^{n-1} (v+1)^k (\frac{P_v}{p_v})^k (A_{n-v-1}^{\alpha-1})^k |\Delta \in_v | \{\sum_{v=1}^{n-1} |$$
$$\times \Delta \in_v |\}^{k-1}$$

$$= O(1) \sum_{v=1}^{m} v^k (\frac{P_v}{p_v})^k |\Delta \in_v ||\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{(A_{n-v-1}^{\alpha-1})^k}{n(A_n^{\alpha})^k}$$

$$= O(1) \sum_{v=1}^{m} v^k (\frac{P_v}{p_v})^k |\Delta \in_v ||\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1} \frac{(n-v)^{k\alpha-k}}{n^{1+k\alpha}}$$

$$= O(1) \sum_{v=1}^{m} (\frac{P_v}{p_v})^k |\Delta \in_v ||\Delta T_{v-1}|^k$$

$$= O(1) \sum_{v=1}^{\infty} (\frac{P_v}{p_v})^k |\Delta T_{v-1}|^k | \sum_{n=v}^{\infty} \Delta^2 \in_n |$$

$$= O(1) \sum_{v=1}^{\infty} (\frac{P_v}{p_v})^k |\Delta T_{v-1}|^k \sum_{n=v}^{\infty} |\Delta^2 \in_n |$$

$$= O(1) \sum_{n=1}^{\infty} |\Delta^2 \in_n | \sum_{v=1}^{n} (\frac{P_v}{p_v})^k |\Delta T_{v-1}|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{P_n}{p_n}|\Delta^2 \in_n|\sum_{v=1}^{n}(\frac{P_v}{p_v})^{k-1}|\Delta T_{v-1}|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{P_n}{p_n}|\Delta^2 \in_n|\lambda_n^k$$

For $t_{n,3}^{\alpha}$, we have to consider the two cases $0 < \alpha < 1$ & $\alpha \geq 1$. For $0 < \alpha < 1$, we obtain

$$\sum_{n=2}^{m+1}\frac{1}{n}|t_{n,3}^{\alpha}|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{1}{n(A_n^{\alpha})^k}\sum_{v=1}^{n-1}(v+1)^k(\frac{P_v}{p_v})^k|\Delta_v A_{n-v}^{\alpha-1}||\in_v|^k|\Delta T_{v-1}|^k\{\sum_{v=1}^{n-1}|\Delta_v A_{n-v}^{\alpha-1}|\}^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+k\alpha}}\sum_{v=1}^{n-1}v^k(\frac{P_v}{p_v})^k(n-v)^{\alpha-2}|\in_v|^k|\Delta T_{v-1}|^k$$

$$\times \{\sum_{v=1}^{n-1}(n-v)^{\alpha-2}\}^{k-1} \qquad (i)$$

$$= O(1)\sum_{v=1}^{m}v^k(\frac{P_v}{p_v})^k|\in_v|^k|\Delta T_{v-1}|^k\sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{1+k\alpha}}$$

$$= O(1)\sum_{v=1}^{m}v^{k-k\alpha-1}(\frac{P_v}{p_v})^k|\in_v|^k|\Delta T_{v-1}|^k$$

$$= O(1)\sum_{v=1}^{\infty}v^{k-k\alpha-1}(\frac{P_v}{p_v})^k|\Delta T_{v-1}|^k\sum_{n=v}^{\infty}|\Delta \in_n|$$

$$= O(1)\sum_{n=1}^{\infty}|\Delta \in_n|\sum_{v=1}^{n}v^{k-k\alpha}\frac{P_v}{vp_v}(\frac{P_v}{p_v})^{k-1}|\Delta T_{v-1}|^k$$

$$= O(1)\sum_{n=1}^{\infty}n^{k-k\alpha-1}\frac{P_n}{p_n}|\Delta \in_n|\lambda_n^k$$

When $\alpha = 1$, $t_{n,3}^{\alpha} = 0$ as $\Delta A_{n-v}^{\alpha-1} = 0$, then we are assuming $\alpha > 1$, and hence

$$\sum_{n=2}^{m+1}\frac{1}{n}|t_{n,3}^{\alpha}|^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{1}{n^{1+k\alpha}}\sum_{v=1}^{n-1}v^k(\frac{P_v}{p_v})^k(n-v)^{\alpha-2}|\in_v|^k|\Delta T_{v-1}|^k\cdot n^{(\alpha-1)(k-1)}$$

$$(\text{ as } \sum_{v=1}^{n-1}(n-v)^{\alpha-2} = O(1)\int_1^{n-1}(n-x)^{\alpha-2}dx = O(n^{\alpha-1}).)$$

$$= O(1)\sum_{v=1}^{m} v^k(\frac{P_v}{p_v})^k| \in_v |^k|\Delta T_{v-1}|^k \sum_{n=v+1}^{m+1}\frac{(n-v)^{\alpha-2}}{n^{k+\alpha}}$$

$$= O(1)\sum_{v=1}^{m}\frac{1}{v}(\frac{P_v}{p_v})^k| \in_v |^k|\Delta T_{v-1}|^k.$$

$$= O(1)\sum_{n=1}^{\infty}\frac{1}{n}\frac{P_n}{p_n}|\Delta \in_n |\lambda_n^k, \quad \text{as in the case of } t_{n,2}^{\alpha}.$$

Finally,

$$\sum_{n=2}^{m+1}\frac{1}{n}|t_{n,5}^{\alpha}|^k = \begin{cases} O(1)\sum_{n=1}^{\infty} n^{k-k\alpha-1}(\frac{P_n}{p_n})^k| \in_n |^k|\Delta T_{n-1}|^k & (0<\alpha<1) \\ O(1)\sum_{n=1}^{\infty}\frac{1}{n}(\frac{P_n}{p_n})^k| \in_n |^k|\Delta T_{n-1}|^k & (\alpha\geq 1) \end{cases}$$

$$= \begin{cases} O(1)\sum_{n=1}^{\infty} n^{k-k\alpha-1}\frac{P_n}{p_n}|\Delta \in_n |\lambda_n^k & (0<\alpha<1) \\ O(1)\sum_{n=1}^{\infty}\frac{1}{n}\frac{P_n}{p_n}|\Delta \in_n |\lambda_n^k & (\alpha\geq 1), \end{cases}$$

as in the case of $t_{n,3}^{\alpha}$.

This completes the proof of the Theorem.

References

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Dept. of Applied Sciences, College of Technological Studies, P. O. Box 42325, 70654 Kuwait.