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COINCIDENCE THEOREMS AND MATCHING THEOREMS

HWEI-MEI KO AND KOK-KEONG TAN(*)

Abstract. Two coincidence theorems of Ky Fan are first slightly generalized. As applications, new matching theorems are obtained, one of which has several equivalent forms, including the classical Knaster-Kuratowski-Mazurkiewicz theorem.

1. Introduction

For a non-empty set X, we shall denote by 2^X the collection of all non-empty subsets of X. If X is a topological space and $A \subset X$, we shall denote by $\overline{A}$ the closure of A and by ∂A the boundary of A. If E is a topological vector space, we shall denote by E' the vector space of all continuous linear functionals on E and by $\langle w, x\rangle$ for $w \in E'$ and $x \in E$ the pairing between E' and E. If $A \subset E$, $co(A)$ (respectively, $\overline{co}(A)$) denotes the convex (respectively, closed convex) hull of A. Suppose $X \subset E$ is non-empty; then a map $f : X \to 2^E$ is said to be upper hemi-continuous ([1, p.68]; see also [2, p.133]) if for each $\phi \in E'$ and for each real number λ, the set $\{x \in X : \sup_{u\in f(x)} Re\langle \phi, u\rangle < \lambda\}$ is open in X. We note that every upper semi-continuous map is upper hemi-continuous and the sum of two upper hemi-continuous maps is again upper hemi-continuous.

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For relations among upper semi-continuity, upper demi-continuity [4] and upper hemi-continuity, we refer to Shih-Tan [11, Propositions 1 and 2 and Examples 1 and 2].

In this paper, we first slightly generalize coincidence theorems of Fan [5, Theorems 9 and 10] and a fixed point theorem of Shin-Tan [11, Theorem 4]. As applications, new matching theorems are obtained, some of which generalize those of Fan [5, Theorems 2 and 11]. Finally, from one of our matching theorems, we deduce several equivalent results, one of which is equivalent to the classical Knaster-Kuratowski-Mazurkiewicz theorem [9].

2. Coincidence Theorems

First we shall state without proof the following result which can be proved by slightly modifying the proofs of Theorem 9 of Fan in [5] and Lemma 1.2 of Ko-Tan in [10]:

Theorem 1. *Let X be a paracompact convex set in a locally convex Hausdorff topological vector space E, X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $f, g : X \to 2^E$ be upper hemi-continuous such that*

(a) *For each $x \in X$, $f(x)$ and $g(x)$ are closed convex, at least one of which is compact.*

(b) *For any $x \in K \bigcap \partial X$ and $\phi \in E'$ with $Re\phi(x) \leq Re\phi(y)$ for all $y \in X$, there exist $u \in f(x)$ and $v \in g(x)$ such that $Re\phi(u) \geq Re\phi(v)$.*

(c) *For any $x \in X \setminus K$ and $\phi \in E'$ with $Re\phi(x) \leq Re\phi(y)$ for all $y \in X_0$, there exist $u \in f(x)$ and $v \in g(x)$ such that $Re\phi(u) \geq Re\phi(v)$.*

Then there exists a point $\hat{x} \in X$ such that $f(\hat{x}) \bigcap g(\hat{x}) \neq \phi$.

The following result is a consequence of Theorem 1 and is a generalization of Theorem 10 of Fan in [5] and Theorem 4 of Shih-Tan in [11]:

Theorem 2. *Let X be a paracompact convex set in a locally convex Haus-*

dorff topological vector space E, X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $f, g : X \to 2^E$ be upper hemi-continuous such that

(a) *For each $x \in X$, $f(x)$ and $g(x)$ are closed convex at least one of which is compact.*

(b) *For each $x \in K \bigcap \partial X$, $f(x) - g(x)$ meets $\overline{\bigcup_{\lambda>0} \lambda(X - x)}$.*

(c) *For each $x \in X \setminus K$, $f(x) - g(x)$ meets $\overline{\bigcup_{\lambda>0} \lambda(X_0 - x)}$.*

Then there exists a point $\hat{x} \in X$ such that $f(\hat{x}) \bigcap g(\hat{x}) \neq \phi$.

Proof. Let $x \in K \bigcap \partial X$ and $\phi \in E'$ be such that

$$Re\phi(x) \leq Re\phi(y) \quad \text{for all} \quad y \in X. \tag{1}$$

As $f(x) - g(x)$ meets $\overline{\bigcup_{\lambda>0} \lambda(X - x)}$, let $u \in f(x)$, $v \in g(x)$, $(\lambda_\alpha)_{\alpha \in \Gamma}$ be a net in $(0, \infty)$ and $(x_\alpha)_{\alpha \in \Gamma}$ be a net in X such that $\lambda_\alpha(x_\alpha - x) \to u - v$; it follows that

$$\lambda_\alpha \phi(x_\alpha - x) = \phi(\lambda_\alpha(x_\alpha - x)) \to \phi(u - v) = \phi(u) - \phi(v).$$

By (1), for each $\alpha \in \Gamma$, $Re\phi(x) \leq Re\phi(x_\alpha)$, so that $Re\phi(u) \geq Re\phi(v)$. Thus the condition (b) in Theorem 1 is satisfied.

Next let $x \in X \setminus K$ and $\phi \in E'$ be such that

$$Re\phi(x) \leq Re\phi(y) \quad \text{for all} \quad y \in X_0. \tag{2}$$

As $f(x) - g(x)$ meets $\overline{\bigcup_{\lambda>0} \lambda(X_0 - x)}$, let $u \in f(x)$, $v \in g(x)$, $(\lambda_\alpha)_{\alpha \in \Gamma}$ be a net in $(0, \infty)$ and $(x_\alpha)_{\alpha \in \Gamma}$ be a net in X_0 such that $\lambda_\alpha(x_\alpha - x) \to u - v$; it follows from (2) that $Re\phi(u) \geq Re\phi(v)$. Thus the condition (c) in Theorem 1 is also satisfied.

Therefore by Theorem 1, there exists $\hat{x} \in X$ such that $f(\hat{x}) \bigcap g(\hat{x}) \neq \phi$.

We note that Theorem 2 remains valid if in the union $\bigcup_{\lambda>0}$ in both conditions (b) and (c) "$\lambda > 0$" is replaced by "$\lambda < 0$".

When $g(x) = x$ for all $x \in X$, the coincidence Theorem 2 becomes the following very general fixed point theorem which generalizes a fixed point theroem of Halpern [7, Theorem 2] which in turn generalizes Fan-Glicksberg's infinite dimensional generalization [3,6] of the Kakutani fixed point theorem [8]:

Theorem 3. *Let X be a paracompact convex set in a locally convex Hausdorff topological vector space E, X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $f : X \to 2^E$ be an upper hemi-continuous map such that*

(a) *For each $x \in X$, $f(x)$ is closed and convex.*

(b) *For each $x \in K \bigcap \partial X$, $f(x) \bigcap [x + \overline{\bigcup_{\lambda>0} \lambda(X - x)}] \neq \phi$.*

(c) *For each $x \in X \setminus K$, $f(x) \bigcap [x + \overline{\bigcup_{\lambda>0} \lambda(X_0 - x)}] \neq \phi$.*

Then there exists a point $\hat{x} \in X$ such that $\hat{x} \in f(\hat{x})$.

As is noted earlier, Theorem 3 remains valid if in the union $\bigcup_{\lambda>0}$ in both conditions (b) and (c) "$\lambda > 0$" is replaced by "$\lambda < 0$" and the result so formulated generalizes Theorem 3 in [7].

3. Matching Theorems

As an application of Theorem 2, we have the following matching theorem for closed coverings of a convex set:

Theorem 4. *Let X be a paracompact convex subset of a locally convex Hausdorff topological vector space E, X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be two locally finite families of closed subsets of X such that*

$$\bigcup_{i \in I} A_i = \bigcup_{j \in J} B_j = X.$$

Let $\{C_i : i \in I\}$ and $\{D_j : j \in J\}$ be two families of non-empty subsets of E such that any finite union of the C_i's is contained in a compact convex subset of

E. Let $s : X \to 2^E$ be upper hemi-continuous such that each $s(x)$ is a compact convex set. Suppose that for each point $x \in (K \bigcap \partial X) \bigcup (X \setminus K)$, there exist $i \in l$ and $j \in J$ sucht that

(i) $x \in A_i \bigcap B_j$,

(ii) $\overline{co}(C_i + s(x)) - \overline{co}(D_j)$ *meet* $\begin{cases} \overline{\bigcup_{\lambda>0} (X - x)}, & \text{if } x \in K \cap \partial X, \\ \overline{\bigcup_{\lambda>0} (X_0 - x)}, & \text{if } x \in X \setminus K. \end{cases}$

Then there exist two non-empty finite sets $l_0 \subset l$ and $J_0 \subset J$ and a point $\hat{x} \in X$ such that

(a) $\hat{x} \in (\bigcap_{i \in l_0} A_i) \bigcap (\bigcap_{j \in J_0} B_j)$,

(b) $\overline{co}(\bigcup\{C_i : i \in l_0\}) + s(\hat{x})$ *meets* $\overline{co}(\bigcup\{D_j : j \in J_0\})$.

Proof. For each $x \in X$, let

$$l(x) = \{i \in l : x \in A_i\}, J(x) = \{j \in J : x \in B_j\};$$

then $l(x)$ and $J(x)$ are non-empty and finite as $\bigcup_{i \in l} A_i = \bigcup_{j \in J} B_j = X$ and $\{A_i : i \in l\}$ and $\{B_j : j \in J\}$ are locally finite. Define $f, g, h : X \to 2^E$ by

$$\begin{aligned} f(x) &= \overline{co}(\cup\{C_i + s(x) : i \in l(x)\}), \\ g(x) &= \overline{co}(\cup\{D_j : j \in J(x)\}), \\ h(x) &= \overline{co}(\cup\{C_i : i \in l(x)\}). \end{aligned}$$

By hypotheses, for each $x \in X$, $h(x)$ and $s(x)$ are compact convex so that $f(x) = h(x) + s(x)$ is also compact convex. Since $\{A_i : i \in l\}$ is a locally finite family of closed subsets of X, for each $x \in X$, the set $U(x) = X \setminus \bigcup_{i \notin l(x)} A_i$ is an open neighborhood of x in X; note then whenever $y \in U(x)$, $y \notin A_i$ for each $i \notin l(x)$ so that $l(y) \subset l(x)$ and therefore $h(y) \subset h(x)$. This shows that h is upper semi-continuous and hence $f = h + s$ is upper hemi-continuous. Similarly we can show that g is upper semi-continuous (and hence upper hemi-continuous) on X. Thus the condition (a) of Theorem 2 is satisfied. By (i) and (ii), the conditions (b) and (c) of Theorem 2 are also satisfied. By Theorem 2, there exists $\hat{x} \in X$

such that $f(\hat{x})\bigcap g(\hat{x}) \neq \phi$. Let $l_0 = l(\hat{x})$ and $J_0 = J(\hat{x})$, then l_0 and J_0 are non-empty and finite and the conclusions of the theorem hold.

The proof of the above theorem is a modification of Theorem 11 of Fan in [5] and of Theorem 1 of Shih-Tan in [11]. Theorem 4 generalizes Theorem 11 and hence also Theorem 12 of Fan in [5]. The following result is an easy consequence of Theroem 4:

Theorem 5. *Let X be a paracompact convex subset of a locally convex Hausdorff topological vector space E, X_0 be a non-empty compact convex subset of X and K be a non-empty compact subset of X. Let $\{A_i : i \in l\}$ and $\{B_j : j \in J\}$ be two locally finite families of closed subsets of X such that*

$$\bigcup_{i\in l} A_i = \bigcup_{j\in J} B_j = X.$$

Let $\{C_i : i \in l\}$ and $\{D_j : j \in J\}$ be two families of non-empty subsets of E such that any finite union of C_i's is contained in a compact convex subset of E. Suppose that for each point $x \in (K\bigcap \partial X)\bigcup(X\setminus K)$, there exist $i \in l$ and $j \in J$ such that

(i) $x \in A_i\bigcap B_j$,

(ii) $\overline{co}(C_i) - \overline{co}(D_j)$ *meets* $\begin{cases} x + \overline{\bigcup_{\lambda>0}\lambda(X-x)}, & \text{if } x \in K\bigcap\partial X, \\ x + \overline{\bigcup_{\lambda>0}\lambda(X_0-x)}, & \text{if } x \in X\setminus K. \end{cases}$

Then there exist non-empty finite sets $l_0 \subset l$ and $J_0 \subset J$ such that

$$\Big(\bigcap_{i\in l_0} A_i\Big)\cap\Big(\bigcap_{j\in J_0} B_j\Big)\cap\Big(\overline{co}\Big(\bigcup_{i\in l_0} C_i\Big) - \overline{co}\Big(\bigcup_{j\in J_0} D_j\Big)\Big) \neq \phi.$$

Proof. Let $s : X \to 2^E$ be defined by $s(x) = -x$ for all $x \in X$. Then all hypotheses of Theorem 4 are satisfied so that there exist non-empty finite sets $l_0 \subset l$ and $J_0 \subset J$ and a point $\hat{x} \in X$ such that

(a) $\hat{x} \in (\bigcap_{i\in l_0} A_i)\bigcap(\bigcap_{j\in J_0} B_j)$,

(b) $(\overline{co}(\bigcup_{i\in l_0} C_i) - \hat{x})\bigcap\overline{co}(\bigcup_{j\in J_0} D_j) \neq \phi$;

it follows that

$$\hat{x} \in (\bigcap_{i\in I_0} A_i) \cap (\bigcap_{j\in J_0} B_j) \cap (\overline{co}(\bigcup_{i\in I_0} C_i) - \overline{co}(\bigcup_{i\in J_0} D_j)).$$

As an application of Theorem 5, we have

Theorem 6. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $y_1, \ldots, y_n$ be points in Y, $z_1, \ldots, z_m$ be points in E, $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ be closed subsets of Y such that $\bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} B_j = Y$. Suppose whenever $1 \leq i_1 < \ldots < i_k \leq n$ with $1 \leq k < n$ and whenever $x \in co\{y_{i_1}, \ldots, y_{i_k}\}$, there exists $j \in \{1, \ldots, m\}$ such that $x \in B_j$ and $x + z_j \in co\{y_{i_1}, \ldots, y_{i_k}\}$. Then there exist two non-empty subfamilies $\mathcal{G}$ of $\{1, \ldots, n\}$ and $\mathcal{H}$ of $\{1, \ldots, m\}$ such that*

$$(\bigcap_{i\in\mathcal{G}} A_i) \cap (\bigcap_{j\in\mathcal{H}} B_j) \cap [(co\{y_i : i \in \mathcal{G}\}) - (co\{z_j : j \in \mathcal{H}\})] \neq \phi.$$

Proof. Let $X = X_0 = K = co\{y_1, \ldots, y_n\}$, $I = \{1, \ldots, n\}$, $J = \{1, \ldots, m\}$ and

$$C_i = \{y_i\},\ \tilde{A}_i = A_i \cap X \quad \text{for each} \quad i \in I,$$
$$D_j = \{z_j\},\ \tilde{B}_j = B_j \cap X \quad \text{for each} \quad j \in J,$$

We shall show that for each $x \in \partial X$, there exist $i \in I$ and $j \in J$ such that

(i) $x \in \tilde{A}_i \bigcap \tilde{B}_j$,

(ii) $y_i - z_j \in x + \bigcup_{\lambda>1} \lambda(X - x)$.

Indeed, let $x \in \partial X$; without loss of generality we may assume that $x = \sum_{k=1}^{s} a_k y_k$ where $1 \leq s < n$ and $a_k > 0$ for each $k = 1, \ldots, s$ with $\sum_{k=1}^{s} a_k = 1$. Thus by hypothesis, there exist $i \in I$ and $j \in J$ such that $x \in \tilde{A}_i \bigcap \tilde{B}_j$ and $x + z_j \in co\{y_1, \ldots, y_s\}$. Let $x + z_j = \sum_{k=1}^{s} b_k y_k$ where $b_k \geq 0$ for each $k = 1, \ldots, s$ and $\sum_{k=1}^{s} b_k = 1$. As $a_k > 0$ for all $k = 1, \ldots, s$, we can choose $\lambda > 1$ such that $\lambda a_k - b_k \geq 0$ for all $k = 1, \ldots, s$. As $\sum_{k=1}^{s} (\lambda a_k - b_k) = \lambda(\sum_{k=1}^{s} a_k) - (\sum_{k=1}^{s} b_k) = \lambda - 1$,

$\frac{1}{\lambda-1}\sum_{k=1}^{s}(\lambda a_k - b_k)y_k$ is in X. It follows that

$$\frac{1}{\lambda}(y_i - z_j - x) + x = \frac{1}{\lambda}(y_i - \sum_{k=1}^{s} b_k y_k) + (\sum_{k=1}^{s} a_k y_k)$$

$$= \frac{1}{\lambda}[1 \cdot y_i + (\lambda - 1) \cdot \frac{1}{\lambda - 1}\sum_{k=1}^{s}(\lambda a_k - b_k)y_k] \in X$$

so that $y_i - z_j \in x + \lambda(X - x) \subset x + \bigcup_{\lambda > 1} \lambda(X - x)$.

Since X is a compact convex set in a Euclidean space, Theorem 5 implies there exists two non-empty subfamilies $\mathcal{G}$ of $\{1, \ldots, n\}$ and $\mathcal{H}$ of $\{1, \ldots, m\}$ such that

$$(\bigcap_{i \in \mathcal{G}} \tilde{A}_i) \cap (\bigcap_{j \in \mathcal{H}} \tilde{B}_j) \cap (co\{y_i : i \in \mathcal{G}\} - co\{z_j : j \in \mathcal{H}\}) \neq \phi;$$

$$(\bigcap_{i \in \mathcal{G}} A_i) \cap (\bigcap_{j \in \mathcal{H}} B_j) \cap (co\{y_i : i \in \mathcal{G}\} - co\{z_j : j \in \mathcal{H}\}) \neq \phi.$$

This completes the proof.

4. Equivalent forms of the Classical Knaster-Kuratowski-Mazurkiewicz Theorem

By taking $z_j = 0$ and $B_j = Y$ for all $j = 1, \ldots, m$ in Theorem 6, we have the following very general matching theorem due to Fan [5, Theorem 2] where he has given two basically different proofs:

Theorem 7A. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $y_1, \ldots, y_n$ be points in Y, $A_1, \ldots, A_n$ be closed subsets of Y such that $\bigcup_{i=1}^{n} A_i = Y$. Then there exists a non-empty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ such that*

$$(\bigcap_{j=1}^{n} A_{i_j}) \cap co\{y_{i_j} : j = 1, \ldots, k\} \neq \phi.$$

Now we shall state the following Theorems 7B, 7C, 7D and 7E which are all equivalent to Theorem 7A:

Theorem 7B. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $y_1 \dots, y_n$ be points in Y and $B_1, \dots, B_n$ be open subsets of Y such that $\bigcup_{i=1}^{n} B_i = Y$. Then there exists a non-empty subset $\{i_i, \dots, i_k\}$ of $\{1, \dots, n\}$ such that*

$$(\bigcap_{j=1}^{k} B_{i_j}) \cap (co\{y_{i_j} : j = 1, \dots, k\}) \neq \phi.$$

Theorem 7C. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $y_1, \dots, y_n$ be points in Y and $A_1, \dots, A_n$ be closed subsets of Y such that $co\{y_i : i \in l\} \subset \bigcup_{i \in l} A_i$ for each subset l of $\{1, \dots, n\}$. Then $\bigcap_{i=1}^{n} A_i \neq \phi$.*

Theorem 7D. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $y_1, \dots, y_n$ be points in Y and $B_1, \dots, B_n$ be open subsets of Y such that $co\{y_i : i \in l\} \subset \bigcup_{i \in l} B_i$ for each subset l of $\{1, \dots, n\}$. Then $\bigcap_{i=1}^{n} B_i \neq \phi$.*

Theorem 7E. *Let Y be a non-empty convex set in a Hausdorff topological vector space E, $\{y_i : i \in l\}$ be a family of points in Y which is contained in a compact convex subset of Y and $\{O_i : i \in l\}$ be a family of open subsets of Y such that $\bigcup_{i \in I} O_i = Y$. Then there exists a non-empty finite subset l_0 of l such that*

$$(\bigcap_{i \in l_0} O_i) \cap (co\{y_i : i \in l_0\}) \neq \phi.$$

We first remark that Theorem 7A and Theorem 7B (respectively, Theorem 7C and Theorem 7D) are dual statements of each other in the sense that the words "closed" and "open" are interchangeable. However, a dual statement of Theorem 7E does not hold even if the set Y is compact as the following simple example illustrates:

Example. Let $Y = l = [0,1]$. For each $i \in l$ with $i \neq \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, let $A_i = \{1 - i\}$ and let $A_{\frac{1}{3}} = \{\frac{1}{2}\}$, $A_{\frac{1}{2}} = \{\frac{2}{3}\}$, $A_{\frac{2}{3}} = \{\frac{1}{3}\}$. Then $\{A_i : i \in l\}$ is a family of closed subsets of Y such that

(a) $\bigcup_{i \in l} A_i = Y$;

(b) $i \notin A_i$ for each $i \in l$;

(c) $A_i \bigcap A_j = \phi$ if $i \neq j$.

Thus there does not exist a non-empty finite subset l_0 of l such that

$$(\bigcap_{i \in l_0} A_i) \cap (co\{i : i \in l_0\}) \neq \phi.$$

We next remark that Thoerem 7C is equivalent to the classical Knaster-Kuratowski-Mazurkiewicz theorem [9] while Theorem 7D is equivalent to Corollary 1 in [12]. However the formulations of Theorems 7B and 7E appear to be new. Thus once we have established the equivalence of Theorems 7A, 7B, 7C, 7D and 7E, our matching Theorem 6 and hence also Theorems 4 and 5, are all generalizations of the classical Knaster-Kuratowski-Mazurkiewicz theorem.

It is easy to see that Theorems 7B and 7E imply each other. Before we proceed to establish the equivalence of Theorems 7A, 7B, 7C and 7D, we observe that by replacing A_i's and B_i's by $A_i \bigcap co\{y_1, \ldots, y_n\}$ and $B_i \bigcap co\{y_1, \ldots, y_n\}$ respectively, *we may assume without loss of generality that* $Y = co\{y_i, \ldots, y_n\}$ *and is therefore also compact in some Euclidean space.* We shall only prove that Theorem 7A $\Leftrightarrow$ Theorem 7B and Theorem 7B $\Leftrightarrow$ Theorem 7C; the proof of Theorem 7A $\Leftrightarrow$ Theorem 7D follows a similar argument as that of Theorem 7B $\Leftrightarrow$ Theorem 7C and is thus omitted.

Proof of Theorem 7A $\Rightarrow$ Theorem 7B:

For each $z \in Y$, let $H_z = \bigcap\{B_i : i = 1, \ldots, n$ and $z \in B_i\}$; then H_z is an open neighborhood of z in Y so that there exists an open neighborhood U_z of z in Y such that $U_z \subset \overline{U}_z \subset H_z$. Since $Y = \bigcup\{U_z : z \in Y\}$ and Y is compact (remember, we have assumed $Y = co\{y_1, \ldots, y_n\}$, see remark above.) there exist $z_1, \ldots, z_m$ in Y such that $Y = \bigcup\{U_{z_j} : j = 1, \ldots, m\}$. For each $i = 1, \ldots, n$, define

$$A_i = \bigcup(\overline{U}_{z_j} : j = 1, \ldots, m \quad \text{and} \quad H_{z_j} \subset B_i\},$$

then A_i is a closed subset of Y and $A_i \subset B_i$. Clearly $Y = U_{i=1}^n A_i$. Hence by

Theorem 7A, there exists a non-empty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ such that

$$(\bigcap_{j=1}^{k} A_{i_j}) \cap co\{y_{i_j} : j = 1, \ldots, k\} \neq \phi;$$

but then

$$(\bigcap_{j=1}^{k} B_{i_j}) \cap co\{y_{i_j} : j = 1, \ldots, k\} \neq \phi.$$

Proof of Theorem 7B $\Rightarrow$ Theorem 7A:

Fix any positive integer m. For each $i = 1, \ldots, n$, let

$$B_i(m) = \{y \in Y : \text{dist}(y, A_i) < \frac{1}{m}\},$$

(remember we have assumed $Y = co\{y_1, \ldots, y_n\}$ so that Y is contained in some Euclidean space.) then $B_i(m)$ is an open subset of Y and $A_i \subset B_i(m)$. Since $\bigcup_{i=1}^{n} A_i = Y$, we have $\bigcup_{i=1}^{n} B_i(m) = Y$. Hence by Theorem 7B, there exists a non-empty subset l_m of $\{1, \ldots, n\}$ such that

$$(\bigcap_{i \in l_m} B_i(m)) \cap co\{y_i : i \in l_m\} \neq \phi.$$

Since the collection of all non-empty subsets of $\{1, \ldots, n\}$ is finite, there is a sequence $(m_j)_{j=1}^{\infty}$ with $m_j \to \infty$ as $j \to \infty$ and there is a non-empty subset l_0 of $\{1, \ldots, n\}$ such that $lm_j = l_0$ for all $j = 1, 2, \ldots$. For each $j = 1, 2, \ldots$, choose any x_j in $(\bigcap_{i \in l_0} B_i(m_j)) \bigcap (co\ \{y_i : i \in l_0\})$. Since $(x_j)_{j=1}^{\infty}$ is a sequence in the compact set $co\{y_i : i \in l_0\}$, there is a subsequence $(x_{l_j})_{j=1}^{\infty}$ such that $x_{l_j} \to \hat{x}$ for some $\hat{x} \in co\{y_i : i \in l_0\}$. Fix $i \in l_0$. Since

$$\begin{aligned} dist(\hat{x}, A_i) &\leq \text{dist}(\hat{x}, x_{l_j}) + \text{dist}(x_{l_j}, A_i) \\ &< \text{dist}(\hat{x}, x_{l_j}) + \frac{1}{m_{l_j}} \\ &\to 0 \quad \text{as} \quad j \to \infty, \end{aligned}$$

we see that $\hat{x} \in A_i$ as A_i is closed. Therefore $\hat{x} \in \bigcap_{i \in l_0} A_i$ so that

$$(\bigcap_{i \in l_0} A_i) \cap (co\{y_i : i \in l_0\}) \neq \phi.$$

Proof of Theorem 7B ⇒ Theorem 7C:

Suppose $\bigcap_{i=1}^{n} A_i = \phi$; then $Y = Y \setminus \bigcap_{i=1}^{n} A_i = \bigcup_{i=1}^{n}(Y \setminus A_i)$. Let $B_i = Y \setminus A_i$ for each $i = 1, \ldots, n$, then each B_i is open in Y. By Theorem 7B, there exists a non-empty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ such that

$$(\bigcap_{j=1}^{k} B_{i_j}) \cap co\{y_{i_1}, \ldots, y_{i_k}\} \neq \phi.$$

Thus $(Y \setminus \bigcup_{j=1}^{k} A_{i_j}) \bigcap (co\{y_{i_1}, \ldots, y_{i_k}\}) \neq \phi$ so that $co\{y_{i_1}, \ldots, y_{i_k}\} \not\subset \bigcup_{j=1}^{k} A_{i_j}$ which is a contradiction. Therefore we must have

$$\bigcap_{i=1}^{n} A_i \neq \phi.$$

Proof of Theorem 7C ⇒ Theorem 7B:

Suppose the contrary that for any non-empty subset l of $\{1, \ldots, n\}$, $(\bigcap_{i \in l} B_i) \bigcap (co\{y_i : i \in l\}) = \phi$. For each $i = 1, \ldots, n$, let $A_i = Y \setminus B_i$, then A_i is a closed subset of Y. But then $co\{y_i : i \in l\} \subset \bigcup_{i \in l} A_i$ for each non-empty subset l of $\{1, \ldots, n\}$, so that by Theorem 7C, $\bigcap_{i=1}^{n} A_i \neq \phi$. It follows that $\bigcup_{i=1}^{n} B_i \neq Y$ which is a contradiction. Therefore there exists a non-empty subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ such that

$$(\bigcap_{j=1}^{k} B_{i_j}) \cap (co\{y_{i_j} : j = 1, \ldots, k\}) \neq \phi.$$

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Department of Mathematics, National Central University, Chung-Li, Taiwan 32054, Republic of China.

Department of Mathematics, Statistics and Computing Science, Dalhousie Universityy, Halifax, Nova Scotia, B3H 3J5, Canada.