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ON EXTREME POINTS OF A CERTAIN LINEAR SPACE OF LOCALLY UNIVALENT FUNCTIONS

KHALIDA INAYAT NOOR

Abstract. Let $H = (H, \oplus, \odot)$ denote the real linear space of locally univalent normalized functions in the unit disc as defined by Hornich. For $-1 \leq B < A \leq 1$, $k \geq 2$, the classes $V_k[A, B]$ of functions with bounded boundary rotation are introduced and this linear space structure is used to determine the extreme points of the classes $V_k[A, B]$.

1. Introduction

let M be the set of all analytic functions in the unit disc $E = \{z : |z| < 1\}$. Let H denote the subclass of all locally univalent functions f which are normalized by the conditions

$$f(0) = 0, \qquad f'(0) = 1 \tag{1.1}$$

and for which $\arg(f')$ is bounded in E. Here arg denotes that branch of the imaginary part of the logarithm which vanishes at the point 1.

Let $P[A, B]$, $-1 \leq B < A \leq 1$ be the class of functions p analytic in E with $p(0) = 1$ such $p(z)$ is subordinate to $\dfrac{1 + Az}{1 + Bz}$. We note that $P[1, -1] \equiv P = \{p \in M / Re\ p > 0, p(0) = 1\}$.

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Also $K[A,B]$, $-1 \leq B < A \leq 1$, denotes the class of functions $f \in M$ and normalized by (1.1) such that

$$1 + \frac{zf''}{f'} \in P[A,B]. \tag{1.2}$$

$K[1,-1]$ is the class K of convex univalent functions in E and $K[A,B] \subseteq K$.

We define the following:

Definition 1.1. Let $f \in H$ and, for $z \in E$, $-1 \leq B < A \leq 1$,

$$1 + \frac{zf''(z)}{f'(z)} = (\frac{k}{4} + \frac{1}{2})p_1(z) - (\frac{k}{4} - \frac{1}{2})p_2(z), \qquad k \geq 2,$$

where $p_1, p_2 \in P[A,B]$. Then we say that $f \in V_k[A,B]$. The class $V_k[1,-1]$ consists entirely of functions with bounded boundary rotation $k\pi$ for some $k \geq 2$ and $V_2[A,B] \equiv K[A,B]$.

Hornich [1] showed that $(H, \oplus, \odot)$ is a linear space with the operations

$$(f \oplus g)(z) = \int_0^z f'(\xi)g'(\xi)d\xi,$$
$$(\lambda \odot f)(z) = \int_0^z (f'(\xi))^\lambda d\xi.$$

The null element in H is the identity mapping.

We shall show that $V_k[A,B]$ is a linear space under these operations and discuss the extreme points of these classes considered as subsets of H.

2. Main Results

Theorem 2.1. *Let f and $g \in V_k[A,B]$ and let F be defined by*

$$F(z) = \int_0^z (f'(\xi))^\alpha (g'(\xi))^\beta d\xi, \tag{2.1}$$

with $\alpha, \beta \in R^+$ and $\alpha + \beta = 1$. Then $F \in V_k[A,B]$.

Proof. From (2.1), we have

$$\begin{aligned}\frac{(zF'(z))'}{F'(z)} &= \alpha\frac{(zf'(z))'}{f'(z)} + \beta\frac{(zg'(z))'}{g'(z)}\\ &= (\frac{k}{4}-\frac{1}{2})[\alpha p_1(z)+\beta p_3(z)] - (\frac{k}{4}-\frac{1}{2})[\alpha p_2(z)+\beta p_4(z)],\end{aligned}$$

$p_i \in p[A,B]$, $i = 1,2,3,4$.

Since $P[A,B]$ is a convex set, see [2]. we have the required result that $F \in V_k[A,B]$.

Theorem 2.2. *Let $p \in P[A,B]$ and $p(z) = 1 + p_1 z + p_2 z^2 + \ldots\ldots$ Then $|p_1| = A - B$ if and only if p has the form*

$$P(z) = \frac{1+Axz}{1+Bxz}, \tag{2.2}$$

where $x \in X = \{x \in C : |X| = 1\}$, $-1 \le B < A \le 1$.

Proof. Since $p \in P[A,B]$, we can write

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)},$$

where w is a Schwarz function.
Thus

$$w(z) = \frac{1-p(z)}{Bp(z)-A},$$

w is analytic with $|w(z)| < 1$ in E and $w(0) = 0$. This gives us

$$|w'(0)| = \left|\frac{(A-B)p'(0)}{(Bp(0)-A)^2}\right| = \left|\frac{1}{(A-B)}p_1\right| = 1.$$

Therefore, from Schwarz lemma, we conclude that w has the form $w = xz$ for some $x \in X$ and this gives us (2.2). Conversely if p is given by (2.2), then it is easy to check that $|p_1| = (A-B)$. This completes the proof.

Remark 2.1. It is easy to see that $p \in P[A,B]$ if there exists a function $h \in P$ such that

$$p(z) = \frac{(1-A)+(1+A)h(z)}{(1-B)+(1+B)h(z)}. \tag{2.3}$$

Using the known result [e.g. 3] determining the extreme points for the class P alongwith (2.3) and theorem 2.2, we easily have the following:

Theorem 2.3. *The extreme points in the class $P[A,B]$ are all the functions of the form*

$$p_e(z) = \frac{1+Axz}{1+Bxz}, \quad x \in X. \tag{2.4}$$

Remark 2.2. From theorem 2.3, Milman theorem [4, p440] and the known [2] fact that $P[A,B]$ is a convex set, we have the following Herglotz integral representation for $p \in P[A,B]$.

$$p(z) = \int_X \frac{1+Axz}{1+Bxz} d\mu, \tag{2.5}$$

where $x \in X$ and μ is a Borel probability measure on X and conversely.

We can also easily deduce that an equivalent condition for $f \in V_k[A,B]$ is that, for $B \neq 0$,

$$\left.\begin{array}{rl} \log f'(z) = & \int_X \log(1-Bxz)^{\frac{A-B}{B}} d\mu, \\ \text{where} & \int_X |d\mu| \leq k. \end{array}\right] \tag{2.6}$$

Also $f \in V_k[A,B]$, $B \neq 0$, implies that

$$|\arg f'(z)| \leq \frac{k}{2}\frac{A-B}{B}\sin^{-1} Br. \tag{2.7}$$

Remark 2.3 Following essentially the same method [5], we can easily show that for every $\alpha, \beta \in R^+$, the class $(\alpha \odot K[A,B] \ominus \beta \odot K[A,B])$ is a compact subset of H whose end points are exactly the functions f of the form

$$f'(z) = \left[\frac{(1+Bxz)^\beta}{(1-Byz)^\alpha}\right]^{\frac{A-B}{B}} \tag{2.8}$$

with some $(x,y) \in X^2$, $x \neq -y$ and $B \neq 0$.

From definition 1.1 and the integral representation (2.6) we immediately have the following.

Theorem 2.4. *For all $k > 2$, $B \neq 0$, we have*

$$V_k[A,B] = (\frac{k}{4} + \frac{1}{2}) \odot K[A,B] \ominus (\frac{k}{4} - \frac{1}{2}) \odot K[A,B].$$

Theorem 2.5. *For all $k > 2$, $B \neq 0$, the class $V_k[A,B]$ is a compact subset of H. A function f is an extreme point of $V_k[A,B]$ with respect to the Hornich linear space structure if and only if*

$$f'(z) = [\frac{(1+Bxz)^{\frac{k}{4}+\frac{1}{2}}}{(1-Byz)^{\frac{k}{4}-\frac{1}{2}}}]^{\frac{A-B}{B}},$$

for some $(x,y) \in X^2$, $x \neq -y$.

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Mathematics Department, College of Science,P.O. Box 2455, King Saud University, Riyadh 11451, Saudi Arabia.