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A MANY VARIABLE GENERALIZATION OF HARDY'S INEQUALITY CONCERNING A SERIES OF TERMS

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Abstract. In the present note we establish a multivariate generalization of the well known Hardy's inequality concerning a series of terms by using a fairly elementary analysis.

1. Introduction

The classical inequality concerning a series of terms due to G. H. Hardy [4] (see, also [5, p.241] and [1,2]) can be stated as follows.

Theorem H. *If $p > 1$, $a_n \geq 0$ and $A_n = a_1 + \ldots + a_n$, then*

$$\sum\left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum a_n^p. \tag{1}$$

The equality holds in (1) *if all the a are zero.*

Despite the many papers that have been written about Hardy's inequality (1), it appears that there is no natural multivariate version of (1) in the literature, see [1-10] and the references therein. In view of the importance of the inequality (1), it is desirable to find many variable analogue of the inequality (1). The main objective of this note is to present a many variable generalization of the

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inequality (1) by using the elementary analysis based on the idea used by E. B. Elliott [3] to obtain a simple proof of Hardy's inequality given in (1).

2. Main Result

In what follows, we let R be the set of real numbers and B be a subset of the n-dimensional Euclidean space R^n defined by $B = \{x \in R^n : \underline{1} \leq x < \infty\}$ where $\underline{1} = (1, \ldots, 1) \in R^n$. For a function $u : B \to R$, we use the following notations

$$\sum_B u(y) = \sum_{y_1=1}^{\infty} \cdots \sum_{y_n=1}^{\infty} u(y_1, \ldots, y_n)$$

and

$$\sum_{B_{\underline{1},x}} u(y) = \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} u(y_1, \ldots, y_n),$$

where $\underline{1} = (1, \ldots, 1) \in B$, $x = (x_1, \ldots, x_n) \in B$ such that $\underline{1} \leq x$ i.e. $1 \leq x_i$. Throughout this paper without further mention, we assume that all inequalities between vectors are componentwise and all the sums exist on the respective domains of their definitions and agree that the value of any function $u(x_1, \ldots, x_n)$ with any of its component zero is zero.

Our main result is established in the following theorem.

Theorem. *If $p > 1$ is a constant, $f(x) \geq 0$ for $x \in B$ and*

$$A(x) = \sum_{B_{\underline{1},x}} f(y), \quad x \in B, \tag{2}$$

then

$$\sum_B \{\frac{A(x)}{\prod_{i=1}^n x_i}\}^p \leq (\frac{p}{p-1})^{np} \sum_B f^p(x). \tag{3}$$

The equality holds in (3) if $f(x) = 0$ for all x_i, $i = 1, \ldots, n$.

Proof. Let $N = (N_1, N_2, \ldots, N_n) \in B$. From (2) we observe that

$$\sum_{B_{\underline{1},N}} \{\frac{A(x)}{\prod_{i=1}^n x_i}\}^p = \sum_{x_1=1}^{N_1} \cdots \sum_{x_n=1}^{N_n} (\Pi_{i=1}^n x_i)^{-p} \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} f(y_1, \ldots, y_n)\}^p$$

$$= \sum_{x_1=1}^{N_1} \cdots \sum_{x_{n-1}=1}^{N_{n-1}} (\Pi_{i=1}^{n-1} x_i)^{-p} \Big[\sum_{x_n=1}^{N_n} x_n^{-p} \Big\{ \sum_{y_n=1}^{x_n} \cdot \Big\{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} f(y_1, \ldots, y_{n-1}, y_n) \Big\} \Big\}^p \Big]. \tag{4}$$

Define

$$\alpha(x_1, \ldots, x_{n-1}, x_n) = \frac{1}{x_n} \sum_{y_n=1}^{x_n} \Big\{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} f(y_1, \ldots, y_{n-1}, y_n) \Big\}, \tag{5}$$

where $(x_1, \ldots, x_{n-1}, x_n) \in B$. From (5) and using the elementary inequality

$$a^{m+1} + mb^{m+1} \geq (m+1)ab^m, \quad a, b \geq 0, \quad m \geq 1, \tag{6}$$

we observe that

$$\begin{aligned}
&\alpha^p(x_1, \ldots, x_{n-1}, x_n) - (\frac{p}{p-1}) \Big\{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} f(y_1, \ldots, y_{n-1}, x_n) \Big\} \\
&\cdot \alpha^{p-1}(x_1, \ldots, x_{n-1}, x_n) \\
&= \alpha^p(x_1, \ldots, x_{n-1}, x_n) - (\frac{p}{p-1}) \big\{ x_n \alpha(x_1, \ldots, x_{n-1}, x_n) \\
&- (x_n - 1)\alpha(x_1, \ldots, x_{n-1}, x_n - 1) \big\} \alpha^{p-1}(x_1, \ldots, x_{n-1}, x_n) \\
&= [1 - x_n(\frac{p}{p-1})] \alpha^p(x_1, \ldots, x_{n-1}, x_n) \\
&+ (\frac{p}{p-1})(x_n - 1)\alpha(x_1, \ldots, x_{n-1}, x_n - 1)\alpha^{p-1}(x_1, \ldots, x_{n-1}, x_n) \\
&\leq [1 - x_n(\frac{p}{p-1})] \alpha^p(x_1, \ldots, x_{n-1}, x_n) \\
&+ (\frac{p}{p-1})(x_n - 1)[\frac{1}{p} \{ \alpha^p(x_1, \ldots, x_{n-1}, x_n - 1) \\
&+ (p-1)\alpha^p(x_1, \ldots, x_{n-1}, x_n) \}] \\
&= (\frac{1}{p-1}) \big\{ (x_n - 1)\alpha^p(x_1, \ldots, x_{n-1}, x_n - 1) \\
&- x_n \alpha^p(x_1, \ldots, x_{n-1}, x_n) \big\}.
\end{aligned} \tag{7}$$

Now keeping $x_1, \ldots, x_{n-1}$ fixed in (7) and substituting $x_n = 1, 2, \ldots, N_n$ and adding the inequalities we get

$$\begin{aligned}
&\sum_{x_n=1}^{N_n} \alpha^p(x_1, \ldots, x_{n-1}, x_n) \qquad (8)\\
&-\left(\frac{p}{p-1}\right)\sum_{x_n=1}^{N_n}\left\{\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_n-1} f(y_1, \ldots, y_{n-1}, x_n)\right\}\\
&\cdot \alpha^{p-1}(x_1, \ldots, x_{n-1}, x_n)\\
&\le -\left(\frac{1}{p-1}\right)N_n\alpha^p(x_1, \ldots, x_{n-1}, N_n)\\
&\le 0.
\end{aligned}$$

From (8) and using the Hölder's inequality with indices p, $\frac{p}{p-1}$ we have

$$\begin{aligned}
&\sum_{x_n=1}^{N_n} \alpha^p(x_1, \ldots, x_{n-1}, x_n) \qquad (9)\\
&\le \left(\frac{p}{p-1}\right)\sum_{x_n=1}^{N_n}\left\{\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_n-1} f(y_1, \ldots, y_{n-1}, x_n)\right\}\\
&\cdot \alpha^{p-1}(x_1, \ldots, x_{n-1}, x_n)\\
&\le \left(\frac{p}{p-1}\right)\left[\sum_{x_n=1}^{N_n}\left\{\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_n-1} f(y_1, \ldots, y_{n-1}, x_n)\right\}^p\right]^{\frac{1}{p}}\\
&\cdot\left[\sum_{x_n=1}^{N_n} \alpha^p(x_1, \ldots, x_{n-1}, x_n)\right]^{\frac{p-1}{p}}.
\end{aligned}$$

Dividing both sides of (9) by the last factor on the right side of (9) and then raising the result to the pth power, we obtain

$$\begin{aligned}
&\sum_{x_n=1}^{N_n} \alpha^p(x_1, \ldots, x_{n-1}, x_n) \qquad (10)\\
&\le \left(\frac{p}{p-1}\right)^p\sum_{x_n=1}^{N_n}\left\{\sum_{y_1=1}^{x_1}\cdots\sum_{y_{n-1}=1}^{x_n-1} f(y_1, \ldots, y_{n-1}, x_n)\right\}^p.
\end{aligned}$$

From (4), (5) and (10) we get

$$\sum_{B_{\underline{1},N}} \Big\{ \frac{A(x)}{\Pi_{i=1}^{n} x_i} \Big\}^p \tag{11}$$

$$\leq \sum_{x_1=1}^{N_1} \cdots \sum_{x_{n-1}=1}^{N_{n-1}} \Big(\Pi_{i=1}^{n-1} x_i\Big)^{-p} [(\frac{p}{p-1})^p \sum_{x_n=1}^{N_n}$$

$$\cdot \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-1}=1}^{x_{n-1}} f(y_1, \ldots, y_{n-1}, x_n)\}^p]$$

$$= (\frac{p}{p-1})^p \sum_{x_1=1}^{N_1} \cdots \sum_{x_{n-2}=1}^{N_{n-2}} (\Pi_{i=1}^{n-2} x_i)^{-p} \sum_{x_n=1}^{N_n} [\sum_{x_{n-1}=1}^{N_{n-1}} x_{n-1}^{-p}$$

$$\cdot \{ \sum_{y_{n-1}=1}^{x_{n-1}} \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-2}=1}^{x_{n-2}} f(y_1, \ldots, y_{n-2}, y_{n-1}, x_n)\}\}^p].$$

Now by following exactly the same arguments as above we obtain

$$\sum_{x_{n-1}=1}^{N_{n-1}} x_{n-1}^{-p} \{ \sum_{y_{n-1}=1}^{x_{n-1}} \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-2}=1}^{x_{n-2}} f(y_1, \ldots, y_{n-1}, x_n)\}\}^p \tag{12}$$

$$\leq (\frac{p}{p-1})^p \sum_{x_{n-1}=1}^{N_{n-1}} \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-2}=1}^{x_{n-2}} f(y_1, \ldots, y_{n-2}, x_{n-1}, x_n)\}^p.$$

From (11) and (12) we obtain

$$\sum_{B_{\underline{1},N}} \{ \frac{A(x)}{\Pi_{i=1}^{n} x_i} \}^p \tag{13}$$

$$\leq (\frac{p}{p-1})^{2p} \sum_{x_1=1}^{N_1} \cdots \sum_{x_{n-2}=1}^{N_{n-2}} (\Pi_{i=1}^{n-2} x_i)^{-p} \sum_{x_n=1}^{N_n} [\sum_{x_{n-1}=1}^{N_{n-1}}$$

$$\cdot \{ \sum_{y_1=1}^{x_1} \cdots \sum_{y_{n-2}=1}^{x_{n-2}} f(y_1, \ldots, y_{n-2}, x_{n-1}, x_n)\}^p].$$

Continuing in this way, we finally get

$$\sum_{B_{\underline{1},N}} \{ \frac{A(x)}{\Pi_{i=1}^{n} x_i} \}^p \leq (\frac{p}{p-1})^{np} \sum_{B_{\underline{1},N}} f^p(x). \tag{14}$$

By taking $N \to \infty$ i.e. $N_i \to \infty$, $i = 1, \ldots, n$ in (14) we get the desired inequality in (3). The proof of the theorem is complete.

It is interesting to note that in the special case when $n = 1$, the inequality established in (3) reduces to the Hardy's inequality given in (1), written in a slightly different notation. For other generalizations and extensions of Hardy's inequality (1), we refer the interested readers to [1-10] and the references given therein.

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