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ROTARU ALPHA - CONVEX FUNCTIONS

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Abstract. Let $S^*(a,b)$ denote the class of analytic functions f in the unit disc E, with $f(0)=f'(0)-1=0$, satisfying the condition $|(zf'(z)/f(z))-a|<b$, $a\in C$, $|a-1|<b\le Re(a)$, $z\in E$. In this paper the class $S^*(\alpha,a,b)$ of functions f analytic in E, with $f(0)=f'(0)-1=0$, $f(z)f'(z)/z\neq 0$ for z in E and satisfying in E the condition $|J(\alpha,f)-a|<b$, $a\in C$, $|a-1|<b\le Re(a)$, where $J(\alpha,f)=(1-\alpha)(zf'(z)/f(z))+\alpha((zf'(z))'/f'(z))$, α a non-negative real number is introduced. It is proved that $S^*(\alpha,a,b)\subset S^*(a,b)$, if $\alpha>(4b/c)\,|Im(a)|$, $c=(b^2-|a-1|^2)/b$. Further a representation formula for $f\in S^*(\alpha,a,b)$ and an inequality relating the coefficients of functions in $S^*(\alpha,a,b)$ are obtained.

1. Introduction

Let V denote the class of functions f analytic in the unit disc E, with $f(0)=f'(0)-1=0$. A function f of V is said to belong to $S^*(\rho)$, the class of starlike function of order ρ, if $Re(zf'(z)/f(z))>\rho$, $0\le\rho<1$. The class S^* of starlike functions is identified by $S^*(0)\equiv S^*$. In [6] Rotaru investigated properties of the class $S^*(a,b)$ of functions $f\in V$ satisfying $|(zf'(z)/f(z))-a|<b$, $z\in E$, where $a\in C$, $|a-1|<b\le Re(a)$. It is clear that $S^*(a,b)\subset S^*(Re(a)-b)\subset S^*$. Let $K(a,b)$ denote the class of functions f in V for which $zf'\in S^*(a,b)$.

In this paper we combine the notions of Rotaru starlike functions [6] and alpha-convex functions [2] to obtain a new subclass of starlike functions.

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We now introduce the class $S^*(\alpha, a, b)$ of functions $f \in V$ with $f(z)f'(z)/z \neq 0$, $z \in E$, satisfying in E the condition

$$|J(\alpha, f) - a| < b, \quad a \in C, \quad |a-1| < b \leq Re(a),$$

where $J(\alpha, f) = (1-\alpha)(zf'(z)/f(z)) + \alpha(zf'(z))'/f'(z)$ and α a non-negative real number. Functions in $S^*(\alpha, a, b)$ are called Rotaru alpha-convex functions.

Note that $S^*(0, a, b) \equiv S^*(a, b)$ and we investigate a few properties of the class $S^*(\alpha, a, b)$.

2. We require the following lemmas to prove our main theorem

Lemma 2.1. *Let $a \in C$, $|a-1| < b \leq Re(a)$ and $p(z)$ be analytic in E with $p(0) = 1$. Then $|p(z) - a| < b$, $z \in E$, if and only if, there exists a function w analytic in E satisfying $w(0) = 0$, $|w(z)| < 1$ for $z \in E$ such that*

$$p(z) = \frac{1 + A\, w(z)}{1 + \overline{B}\, w(z)}, \quad z \in E,$$

where $A = (b^2 - |a|^2 + a)/b$ and $B = (1-a)/b$.

The proof of this lemma follows by an application of Schwarz's lemma as in Rotaru [6].

From now on A and B will be as in lemma 2.1 and also $c = (b^2 - |a-1|^2)/b$.

Next we have the well known Jack's lemma [1].

Lemma 2.2. *Let $w(z)$ be regular in E with $w(0) = 0$. If there exists a $z_0 \in E$ such that $\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|$ then $z_0 w'(z_0) = kw(z_0)$ for some $k \geq 1$.*

It is well known that all α-convex functions are starlike [2]. We now prove an analogous theorem for the class $S^*(\alpha, a, b)$.

Theorem 2.1. *Let $f(z) \in S^*(\alpha, a, b)$, $\alpha > (4/c)\,|Im(a)|$. Then $f(z) \in S^*(a, b)$.*

Proof. Define an analytic function $w(z)$ in E by

$$\frac{zf'(z)}{f(z)} = \frac{1 + A\, w(z)}{1 + \overline{B}\, w(z)}. \tag{2.1}$$

Clearly $w(0) = 0$ and $w(z) \neq -1/\overline{B}$ in E. In view of lemma 2.1, it sufficies to show that $|w(z)| < 1$. Suppose there exists a $z_0 \in E$ such that $\max\limits_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$, then by lemma 2.2,

$$z_0 w'(z_0) = k w(z_0), \quad k \geq 1. \tag{2.2}$$

From (2.1) and (2.2) we get

$$|J(\alpha, f(z_0)) - a| = b\left|\frac{B + (1 + AB + (\alpha ck)/b) w(z_0) + A w^2(z_0)}{1 + (A + \overline{B}) w(z_0) + A\overline{B} w^2(z_0)}\right|,$$

where $c = (b^2 - |a-1|^2)/b > 0$. Now $|J(\alpha, f(z_0)) - a| > b$, provided

$$|B + (1 + AB + (\alpha ck)/b) w(z_0) + A w^2(z_0)|^2 > |1 + (A + \overline{B}) w(z_0) + A\overline{B} w^2(z_0)|^2. \tag{2.3}$$

Condition (2.3) reduces to the following:

$$(\alpha ck)/2b + 1 + Re(AB) + Re[(A + \overline{B}) w(z_0)] > 0$$

or equivalently

$$\frac{\alpha c(k-1)}{2b} + 1 + Re(A)\, Re(B) - Im(A)\, Im(B) + Re(A + \overline{B})\, Re(w(z_0))$$
$$- Im(A + \overline{B})\, Im(w(z_0)) + \frac{\alpha c}{2b} > 0. \tag{2.4}$$

Since $\frac{\alpha c}{2b}(k-1) \geq 0$ and $Im(A)\, Im(B) < 0$, (2.4) holds provided $\frac{\alpha c}{2b} \pm Im(A + \overline{B}) > 0$ and $1 + Re(A)\, Re(B) \pm Re(A + \overline{B}) > 0$. Now, since $\alpha > (4/c)\, |Im(a)|$ we have

$$\frac{\alpha c}{2b} \pm Im(A + \overline{B}) > 0$$

and

$$1 + Re(A)\, Re(B) \pm Re(A + \overline{B}) = (1 \pm Re(A))(1 \pm Re(B)) \geq 0,$$

since $|A| \leq 1$ and $|B| < 1$.

This means that $f(z_0) \in S^*(\alpha, a, b)$, a contradiction. Thus the proof is complete.

Corollary 2.1. *Let $f(z) \in S^*(\alpha, a, b)$, $\alpha > 0$. Then $f(z) \in S^*(a, b)$.*

The above corollary follows by taking $a = \overline{a}$ in theorem 2.1.

Corollary 2.2. *Let $\alpha > 0$, $b > 1/2$ and $f(z) \in S^*(\alpha, b, b)$. Then $f(z) \in S^*(b, b)$.*

The above result is obtained if, in corollary 2.1, we take $a = b$, $b > 1/2$.

Remark. Corollary 2.2 improves a result of Miller, Mocanu and Reade [4] who proved it when $b \geq 1$.

Let us choose $a = (1 + \rho - 2\rho\beta)/2(1 - \beta)$ and $b = (1 - \rho)/2(1 - \beta)$, where $0 \leq \rho < 1$ and $0 < \beta \leq 1$. Then the undermentioned corollary follows now from corollary 2.1.

Corollary 2.3. *Let $\alpha > 0$, $0 \leq \rho < 1$, $0 < \beta \leq 1$ and $f \in V$ with $f(z)f'(z)/z \neq 0$ for z in E. Then*

$$\left|\frac{(zf'(z)/f(z)) - 1}{2\beta((zf'(z)/f(z)) - \rho) - ((zf'(z)/f(z)) - 1)}\right| < 1, \quad \text{for } z \text{ in } E.$$

whenever

$$\left|\frac{J(\alpha, f(z)) - 1}{2\beta(J(\alpha, f(z)) - \rho) - (J(\alpha, f(z)) - 1}\right| < 1, \quad \text{for } z \text{ in } E,$$

where $J(\alpha, f(z)) = (1-\alpha)(zf'(z)/f(z)) + \alpha(zf'(z))'/f'(z)$ and α a non-negative real number.

Remark. For $\rho = (1 - \lambda)/(1 + \lambda)$ and $\beta = (1 + \lambda)/2$, $0 < \lambda \leq 1$, corollary 2.3 agrees with theorem 1 of Padmanabhan and Bharati [5].

Theorem 2.2. *For $(4/c)\ |Im(a)| < \beta < \alpha$, $S^*(\alpha, a, b) \subset S^*(\beta, a, b)$.*

Proof. Let $f(z) \in S^*(\alpha, a, b)$. Form theorem 2.1 and the identity

$$J(\beta, f(z)) - a = \frac{\beta}{\alpha}[J(\alpha, f(z)) - a] + (1 - \frac{\beta}{\alpha})[zf'(z)/f(z)) - a], \quad \beta < \alpha,$$

we arrive at the desired result $f(z) \in S^*(\beta, a, b)$.

Corollary 2.4. *For* $\alpha \geq 1 > (4/c)\ |Im(a)|$, $S^*(\alpha, a, b) \subset K(a, b)$.

3. In this section, we obtain an important integral representation for the elements of $S^*(\alpha, a, b)$.

Theorem 3.1. *Let* $f(z) \in S^*(\alpha, a, b)$, $\alpha > (4/c)\ |Im(a)|$, *and if for* $(4/c)$ $|Im(a)| < \beta < \alpha$ *we choose the branch of* $[zf'(z)/f(z)]^\beta$ *which is equal to* 1 *when* $z = 0$, *then the function* $F_\beta(z) = f(z)\ [zf'(z)/f(z)]^\beta$ *is in* $S^*(a, b)$.

Proof. A simple calculation yields

$$\frac{zF'_\beta(z)}{F_\beta(z)} = J(\beta, f(z)).$$

Since $f(z) \in S^*(\alpha, a, b)$, by theorem 2.2, we have

$$\left|\frac{zF'_\beta(z)}{F_\beta(z)} - a\right| = |J(\beta, f(z)) - a| < b,$$

for $(4/c)\ |Im(a)| < \beta < \alpha$. Hence $F_\beta(z) \in S^*(a, b)$.

Now we consider the converse problem. Given the function $F(z) \in S^*(a, b)$ and $\alpha > (4/c)\ |Im(a)|$ is the solution

$$f(z) = \{\frac{1}{\alpha}\int_0^z F^{1/\alpha}(t)t^{-1}dt\}^\alpha, \tag{3.1}$$

of the differential equation

$$F(z) = f(z)[zf'(z)/f(z)]^\alpha, \tag{3.2}$$

with boundary condition $f(0) = 0$, a function in $S^*(\alpha, a, b)$? The answer is yes, and our solution provides us with an integral representation formula for functions is $S^*(\alpha, a, b)$.

Theorem 3.2. *Let* $F(z) \in S^*(a, b)$ *and* $\alpha > (4/c)\ |Im(a)|$. *Then* $f(z)$ *defined by* (3.1) *belongs to* $S^*(\alpha, a, b)$.

The proof of this theorem consists of showing that $f(z)$ is well defined, regular in E and is in $S^*(\alpha, a, b)$. The technique is similar to the one employed in [3, Theorem 5] and is omitted.

4. In this section we obtain an inequality for the coefficients of functions in $S^*(\alpha, a, b)$

Theorem 4.1. *Let $f(z) \in S^*(\alpha, a, b)$ and let $s_1 = 0$, $t_1 = 1 - (A/\overline{B})$, $s_m = (1-\alpha)(\beta_m - \alpha_m) + \alpha\gamma_{m-1}$, $t_m = (\alpha - (A/\overline{B}))\alpha_m + (1-\alpha)\beta_m + \alpha\gamma_{m-1}$, $m = 2, 3, 4, \ldots$, where α_m, β_m and γ_m are defined by*

$$\begin{aligned} \alpha_m &= \sum_{k=1}^{m} (m-k+1) a_k a_{m-k+1}, \\ \beta_m &= \sum_{k=1}^{m} k(m-k+1) a_k a_{m-k+1}, \qquad (4.1) \\ \gamma_m &= \sum_{k=1}^{m} k(k+1) a_{k+1} a_{m-k+1}. \end{aligned}$$

Then the coefficients a_n satisfy the following inequality:

$$\sum_{m=1}^{n} |s_m|^2 \leq |B|^2 \sum_{m=1}^{n} |t_m|^2, \quad n = 2, 3, 4, \ldots$$

Proof. Since $f(z) \in S^*(\alpha, a, b)$,

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[\frac{(zf'(z))'}{f'(z)}\right] = \frac{1 + A\, w(z)}{1 + \overline{B}\, w(z)},$$

where w is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E. This gives

$$\begin{aligned} &(1-\alpha)[z(f'(z))^2 - f(z)f'(z)] + \alpha z f(z) f''(z) \qquad (4.2) \\ &= -\overline{B}w(z)[\alpha - \frac{A}{\overline{B}})f(z)f'(z) + (1-\alpha)z(f'(z))^2 + \alpha z f(z) f''(z)]. \end{aligned}$$

Given $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we note that

$$f(z)f'(z) = \sum_{m=1}^{\infty} \alpha_m z^m, \; (f'(z))^2 = \sum_{m=1}^{\infty} \beta_m z^{m-1}, \; f(z)f''(z) = \sum_{m=1}^{\infty} \gamma_m z^m,$$

where α_m, β_m and γ_m are defined in (4.1). Thus (4.2) becomes

$$(1-\alpha)\sum_{m=1}^{\infty}(\beta_m-\alpha_m)z^m+\alpha\sum_{m=1}^{\infty}\gamma_m z^{m+1}$$
$$=-\overline{B}w(z)\{(1-\alpha)\sum_{m=1}^{\infty}\beta_m z^m+(\alpha-\frac{A}{\overline{B}})\sum_{m=1}^{\infty}\alpha_m z^m+\alpha\sum_{m=1}^{\infty}\gamma_m z^{m+1}\}$$

which simplifies to

$$\sum_{m=1}^{\infty}s_m z^m = -\overline{B}w(z)\{\sum_{m=1}^{\infty}t_m z^m\}. \tag{4.3}$$

Now from (4.3) we get

$$|\sum_{m=1}^{n}s_m z_m+\sum_{m=n+1}^{\infty}h_m z^m| < |B||\sum_{m=1}^{n-1}t_m z^m|,$$

where h_m's are some complex numbers. This yields

$$\sum_{m=1}^{n}|s_m|^2+\sum_{m=n+1}^{\infty}|h_m|^2\leq|B|^2\sum_{m=1}^{n-1}|t_m|^2$$

or

$$\sum_{m=1}^{n}|s_m|^2\leq|B|^2\sum_{m=1}^{n-1}|t_m|^2.$$

The above result is sharp for the function

$$f_\alpha(z) = \{\frac{1}{\alpha}\int_0^z t^{1/\alpha-1}(1+\frac{1-\overline{a}}{b}t)^{\overline{\frac{c}{\alpha(1-\overline{a})}}}dt\}^\alpha \quad \text{if} \quad a\neq 1$$
$$= \{\frac{1}{\alpha}\int_0^z t^{1/\alpha-1}e^{\frac{bt}{\alpha}}dt\}^\alpha \quad \text{if} \quad a=1.$$

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