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RINGS WITH ASSOCIATORS IN THE LEFT AND MIDDLE NUCLEUS

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Abstract. Let R be a nonassociative ring, N, M and L the left, middle and right nucleus respectively. It is shown that if R a semiprime ring satisfying $(R,R,R) \subset N \cap M$ (resp. $(R,R,R) \subset M \cap L$), then $L \subset M \subset N$ (resp. $N \subset M \subset L$); moreover, R is associative if $((R,R,M),(R,R,R)) = 0$ (resp. $((M,R,R),(R,R,R)) = 0$) or $(M,R) \subset M$; and the Abelian group $(R,+)$ has no elements of order 2. We also prove that if R is a simple ring satisfying char $R \neq 2$, and $(R,R,R) \subset N \cap M$ or $(R,R,R) \subset M \cap L$ then R is associative.

1. Introduction

Let R be a nonassociative ring. We adopt the usual notation for associators and commutators: $(x,y,z) = (xy)z - x(yz)$, $(x,y) = xy - yx$. We shall denote the left nucleus, middle nucleus, right nucleus and nucleus by N, M, L and G respectively. Thus N, M, L and G consists of all elements n such that $(n,R,R) = 0$, $(R,n,R) = 0$, $(R,R,n) = 0$ and $(n,R,R) = (R,n,R) = (R,R,n) = 0$ respectively. A ring R is called prime if the product of any two nonzero ideals is nonzero. R is called semiprime if the only ideal of R which squares to zero is the zero ideal. Kleinfeld [2] weakened Thedy's hypotheses [3] to obtain the following result: If R is a prime ring which satisfies $(x,R,x) \subset N$ and $(R,R) \subset N$, then

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R is either associative or commutative. In the notes [4,5,6,7], we weaken Kleinfeld's hypotheses to obtain the same result. In [1], Kleinfeld proved that if R is a semiprime ring satisfying $(R, R, R) \subset G$ and the Abelian group $(R, +)$ has no elements of order 2, then R is associative. In [8], using Kleinfeld's result we improve his result as follows: If R is a semiprime ring satisfying $(R, R, R) \subset N \cap L$, and the Abelian group $(R, +)$ has no elements of order 2, then R is associative. In [7], using his result we prove that if R is a semiprime ring which satisfies $(R, R, R) \subset N$ and the Abelian group $(R, +)$ has no elements of order 2 and $(N, R) \subset N$ then R is associative. In this note, using his result we show the following: If R is a semiprime ring satisfying $(R, R, R) \subset N \cap M$ (resp. $(R, R, R) \subset M \cap L$), then $L \subset M \subset N$ (resp. $N \subset M \subset L$); moreover, R is associative if $((R, R, M), (R, R, R)) = 0$ (resp. $((M, R, R), (R, R, R)) = 0$) or $(M, R) \subset M$; and the Abelian group $(R, +)$ has no elements of order 2. We also prove that if R is a simple ring satisfying $(R, R, R) \subset N \cap M$ or $(R, R, R) \subset M \cap L$; and char $R \neq 2$ then R is associative.

2. Results

Let R be a nonassociative ring. In every ring one may verify the identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z. \tag{1}$$

Suppose that $n \in N$. Then with $w = n$ in (1) we obtain

$$(nx, y, z) = n(x, y, z) \qquad \text{for all} \quad n \text{ in } N. \tag{2}$$

Assume that $m \in L$. Then with $z = m$ in (1) we get

$$(w, x, ym) = (w, x, y)m \qquad \text{for all} \quad m \text{ in } L. \tag{3}$$

As consequences of (1), (2) and (3), we have that N, M and L are associative subrings of R.

We assume that R satisfies

$$(R, R, R) \subset N \cap M. \tag{$*$}$$

Using (1) and (*) we have

$$w(x,y,z) + (w,x,y)z \in N \cap M. \tag{4}$$

Then with $y \in M$ in (4), we get

$$(R,R,M)R \subset N \cap M. \tag{5}$$

Combining (5) with (*), and using (2) we obtain

$$(R,R,M)(R,R,R) = 0. \tag{6}$$

And with $x \in (R,R,M)$ in (1), and applying (*) and (5) we get

$$R(R,R,M) \subset N. \tag{7}$$

Thus with $z \in M$ in (4), and using (7) we obtain $(R,R,R)M \subset N$. Applying this, (*) and (2) we have

$$(R,R,R)(M,R,R) = 0. \tag{8}$$

Then with $z \in L$ in (4), we get

$$(R,R,R)L \subset N \cap M. \tag{9}$$

Definition 1. Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators.

Note that I may be characterized as all finite sums of associators and right (or left) multiples of associators, as a consequence of (1). Hence we have

$$I = (R,R,R) + (R,R,R)R = (R,R,R) + R(R,R,R). \tag{10}$$

Then with $x \in (R,R,R)$ and $y \in L$ in (1), and using $(*)$ and (9) we obtain $(R(R,R,R),L,R) = 0$. By $(*)$ and (10), this implies $(I,L,R) = 0$. Thus it follows from this, $(*)$, (2) and (10) that

$$(R,R,R)(R,L,R) = 0. \tag{11}$$

Definition 2. Let $B = \{b \in I : (R, R, R)b = 0\}$.

Lemma 1. *If R is a ring satisfying $(R, R, R) \subset N \cap M$, then B is an ideal of R and $I \cdot B = 0$.*

Proof. Assume that $b \in B$, x, y, z, $w \in R$. Then using $(*)$, we have $(x, y, z)(bw) = ((x, y, z)b)w = 0$. Thus $bR \subset B$. On the other hand, applying $(*)$ twice and (1) we obtain $(w, x, y)(zb) = ((w, x, y)z)b = -(w(x, y, z))b = -w((x, y, z)b) = 0$. Hence $Rb \subset B$. At this point we have verified that B is an ideal of R. Obviously, we get $(R, R, R)B = 0$ and $((R, R, R)R)B = 0$. Thus by (10) we obtain $I \cdot B = 0$, as desired.

Lemma 2. *If R is a ring satisfying $(R, R, R) \subset N \cap M$ and $(R, R, R)N \subset M$, then $(R, R, R)(R, N, R) = 0$.*

Proof. Assume that $(R, R, R)N \subset M$. Then with $x \in (R, R, R)$ and $y \in N$ in (1), and using $(*)$ we obtain $(R(R, R, R), N, R) = 0$. Applying this, $(*)$ and (10), we have $(I, N, R) = 0$. Thus this implies $(R, R, R)(R, N, R) = 0$, as desired.

Then with $w \in N$ in (4), we have

$$N(R, R, R) \subset N \cap M. \tag{12}$$

Also with $x \in N$ in (4), we get $(R, N, R)R \subset N \cap M$. Using this, $(*)$ and (2), we obtain

$$(R, N, R)(R, R, R) = 0 \tag{13}$$

Then with $x \in N$ and $y \in (R, R, R)$ in (1), and applying $(*)$ and (12), we have

$$(R, N, (R, R, R)R) = (R, N, (R, R, R))R. \tag{14}$$

Theorem 1. *If R is a semiprime ring satisfying $(R, R, R) \subset N \cap M$, then $G = L \subset M \subset N$; moreover, R is associative if $((R, R, M), (R, R, R)) = 0$ or $(M, R) \subset M$; and the Abelian group $(R, +)$ has no elements of order 2.*

Proof. Using Lemma 1, we have $B^2 = 0$. Thus by semiprimeness of R, we obtain $B = 0$. Applying (8) and (11), we get $(M, R, R) \subset B$ and $(R, L, R) \subset B$. Hence we have $(M, R, R) = (R, L, R) = 0$, and so $G = L \subset M \subset N$.

Assume that $((R, R, M), (R, R, R)) = 0$. Then using this and (6) we obtain $(R, R, R)(R, R, M) = 0$. Thus we get $(R, R, M) \subset B$ and so $(R, R, M) = 0$. Hence $M = L = G$. Thus $(R, R, R) \subset G$. By Kleinfeld's result [1], R is associative.

Assume that M is a Lie ideal of R. Combining this with (12) yields $(R, R, R)N \subset M$. Applying Lemma 2, we have $(R, N, R) \subset B$ and so $(R, N, R) = 0$. Thus $N = M$. Hence $(R, R, R) \subset N$ and $(N, R) \subset N$. By Theorem 1 of [7], R is associative.

Definition 3. Let $A = \{a \in I : a(R, R, R) = 0\}$.

Recall that a ring R is called simple if R is the only nonzero ideal of R. Thus, a simple ring is prime if $R^2 \neq 0$, and a prime ring is semiprime.

Theorem 2. *If R is a simple ring satisfying $(R, R, R) \subset N \cap M$, and char $R \neq 2$, then R is associative.*

Proof. By simplicity of R, we have $R^2 = 0$ or $R^2 = R$; and $I = 0$ or $I = R$. If $R^2 = 0$ or $I = 0$, then R is associative. Assume that $R^2 = R$ and $I = R$.

Thus by (10) and (*), we have $R = R^2 = ((R, R, R) + (R, R, R)R)R = (R, R, R)R + (R, R, R)(R^2) = (R, R, R)R$. Hence using this, (14) and (13), we obtain $(R, N, R) = (R, N, (R, R, R)R) = (R, N, (R, R, R))R = (R, N, (R, R, R))((R, R, R)R) = ((R, N, (R, R, R))(R, R, R))R = 0$. Thus we have $N \subset M$. Combining this with Theorem 1 yields $N = M$. Then with $y \in N$ in (4), we obtain $(R, R, N)R \subset N$. Combining this with (*) yields $(R, R, N)(R, R, R) = 0$. Hence $(R, R, N) \subset A$. Assume that $a \in A$ and $x \in R$. Then $a(R, R, R) = 0$, and using (*) we get $aR = a((R, R, R)R) = (a(R, R, R))R = 0$. Thus $A = \{a \in R : aR = 0\}$. Obviously, $A \subset N$. Because of $N = M$, we obtain $(ax)R = 0$ and $(xa)R = x(aR) = 0$. Hence A is an ideal of R. By simplicity of R and $R^2 = R$, we have $A = 0$. Thus, $(R, R, N) \subset A$ implies

$(R,R,N)=0$. Hence $N=L=G$ by Theorem 1. Thus $(R,R,R)\subset G$. By Kleinfeld's result [1], R is associative. This contradicts to $I=R$. Hence $I=0$. Therefore R is associative.

By symmetry, we can prove the following:

Lemma 3. *If R is a ring satisfying $(R,R,R)\subset M\cap L$, then A is an ideal of R and $A\cdot I=0$.*

Theorem 3. *If R is a semiprime ring satisfying $(R,R,R)\subset M\cap L$, then $G=N\subset M\subset L$; moreover, R is associative if $((M,R,R),(R,R,R))=0$ or $(M,R)\subset M$; and the Abelian group $(R,+)$ has no elements of order 2.*

Theorem 4. *If R is a simple ring satisfying $(R,R,R)\subset M\cap L$, and char $R\neq 2$, then R is associative.*

Using the result of [4], we can improve the result of [5] as follows:

Theorem 5[5]. *If R is a prime ring with $N\neq 0$ satisfying $(x,y,z)+(z,y,x)\in N$ and $(N+NR,R)\subset N$, then R is either associative or commutative.*

Added in proof. In [7], we have proved that if R is a prime ring which satisfies $(x,y,z)+(z,y,x)\in N$, $(N,R)\subset N$ and $((R,R),R,R)\subset N$ then either $(N,R)=0$ or char $R=2$ or R is associative. By adding the hypothesis "$(R,R),R,R)\subset N$" in the Main Theorem of [5], we obtain Thedy's and Kleinfeld's results for the semiprime ring case. This result is in [7]: If R is a semiprime ring which satisfies $(x,y,z)+(z,y,x)\in N$, $(N,R)\subset N$, $(NR,R)\subset N$ and $((R,R),R,R)\subset N$, then R is a subdirect sum of a semiprime associative ring and a semiprime commutative ring.

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