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OPIAL TYPE INEQUALITY IN SEVERAL VARIABLES

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Abstract. In the present note we establish a new integral inequality of the Opial type involving a function of n variables and its partial derivative. A corresponding result on the discrete analogue of the main result is also given.

1. Introduction

The following remarkable inequality

$$\int_0^a \mid z(t)z'(t) \mid dt \leq \frac{a}{2} \int_0^a \mid z'(t) \mid^2 dt, \tag{1}$$

valid for $z(t)$ absolutely continuous function with $z(0) = 0$ was proved in 1960 by Z. Opial [2]. Since that time several proofs of this inequality have been given and analogous inequalities have been established, see [1,p.154-162]. In a very interesting paper [8] G.S. Yang has given a two independent variable analogue of the Opial's inequality given in (1). Further results on Opial type inequalities in two and more independent variables can be found in the recent papers of the present author [3-6]. The aim of the present note is to establish a new integral inequality of the Opial type involving a function of n variables and its partial derivative. The discrete analogue of the main result is also given. The analysis used in the proofs is elementary and the results established provide new estimates on these types of inequalities.

2. Main result

We shall denote by R the set of real numbers and R^n the n-dimensional Euclidean space. Let E be a bounded domain in R^n defined by $E = \prod_{i=1}^n [a_i, b_i]$. For $x_i \in R, x = (x_1, \ldots, x_n)$ denote a variable point in E and $dx = dx_1 \ldots dx_n$. For any continuous real-valued function u defined on E we denote by $\int_E u(x)dx$ the n-fold integral $\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} u(x_1, \ldots, x_n)dx_1 \ldots dx_n$ and for any $x \in E$ we denote by $\int_{E_x} u(y)dy$ the n-fold integral $\int_{a_n}^{x_n} \cdots \int_{a_1}^{x_1} u(y_1, \ldots, y_n)dy_1 \ldots dy_n$. We denote by $F(E)$ the class of continuous functions $u : E \to R$ for which $D_1 \ldots D_n u(x)$ exist and such that

$$u(a_1, x_2, \ldots, x_n) = u(x_1, a_2, x_3 \ldots, x_n) = \cdots = u(x_1, \ldots, x_{n-1}, a_n) = 0, \tag{2}$$

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where $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, \cdots, n$.

Our main result is embodied in the following theorem.

Theorem. 1. *Let $u \in F(E)$. Then*

$$\int_E |\, u(x) \,||\, D_1 \cdots D_n u(x) \,|\, dx \leq \Big(\int_E [(\prod_{i=1}^n (x_i - a_i)) \int_{E_x} |\, D_1 \cdots D_n u(y) \,|^2 \, dy] dx\Big)^{\frac{1}{2}}$$
$$\times \Big(\int_E |\, D_1 \ldots D_n u(x) \,|^2 \, dx\Big)^{\frac{1}{2}}. \tag{3}$$

Proof. For any $u \in F(E)$ we have the following identity

$$u(x) = \int_{E_x} D_1 \ldots D_n u(y) dy. \tag{4}$$

From (4) and by using Schwarz inequality in integral form we observe that

$$|\, u(x) \,| \leq \int_{E_x} |\, D_1 \ldots D_n u(y) \,|\, dy \leq \Big(\prod_{i=1}^n (x_i - a_i)\Big)^{\frac{1}{2}} \Big(\int_{E_x} |\, D_1 \ldots D_n u(y) \,|^2 \, dy\Big)^{\frac{1}{2}}. \tag{5}$$

Now by using Schwarz inequality in integral form we have

$$\int_E |\, u(x) \,||\, D_1 \ldots D_n u(x) \,|\, dx \leq \Big(\int_E |\, u(x) \,|^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_E |\, D_1 \cdots D_n u(x) \,|^2 \, dx\Big)^{\frac{1}{2}}. \tag{6}$$

Using (5) on the right side of (6) we get the desired inequality in (3).This completes the proof of Theorem 1.

Remark 1. It is interesting to observe that

$$\int_E (\prod_{i=1}^n (x_i - a_i)) \Big(\int_{E_x} |\, D_1 \ldots D_n u(y) \,|^2 \, dy\Big) dx \tag{7}$$
$$\leq \Big(\int_E (\prod_{i=1}^n (x_i - a_i)) dx\Big) \Big(\int_E |\, D_1 \ldots D_n u(y) \,|^2 \, dy\Big)$$
$$= \frac{1}{2^n} \prod_{i=1}^n (b_i - a_i)^2 \int_E |\, D_1 \ldots D_n u(x) \,|^2 \, dx.$$

Now using (7) on the right side of (3) we have the following inequality

$$\int_E |\, u(x) \,||\, D_1 \ldots D_n u(x) \,|\, dx \leq \frac{1}{(\sqrt{2})^n} \prod_{i=1}^n (b_i - a_i) \int_E |\, D_1 \cdots D_n u(x) \,|^2 \, dx. \tag{8}$$

If we take $n = 1$ in (8) and denote $a_1 = a, b_1 = b$ and $D_1 u = u', x_1 = x$, then the inequality (8) reduces to the following inequality

$$\int_a^b |\, u(x) \,||\, u'(x) \,|\, dx \leq \frac{(b-a)}{\sqrt{2}} \int_a^b |\, u'(x) \,|^2 \, dx. \tag{9}$$

Here we note that the constant appearing in (9) is larger than the constant obtained in Opial's inequality in (1). The main reason for it is the wastage involved in proving the much more general inequality given in (8). The main step in the proof of inequality (8) there is a significant wastage at (7) which prevents to get the sharp constant in inequality (8) and hence in inequality (9).

3. Discrete analogue

Before starting the discrete analogue of our Theorem 1, we first introduce the basic notations and definitions used in this section. Let $N_0 = \{0, 1, 2, \cdots\}$. For $x = (x_1, \ldots, x_n) \in N_0^n$ and $z : N_0^n \to R$, we define the difference operators

$$\nabla_1 z(x_1, x_2, \cdots, x_n) = z(x_1, x_2, \ldots, x_n) - z(x_1 - 1, x_2, \ldots, x_n),$$
$$\vdots$$
$$\nabla_n z(x_1, \ldots, x_{n-1}, x_n) = z(x_1, \ldots, x_{n-1}, x_n) - z(x_1, \ldots x_{n-1}, x_n - 1).$$

Similarly, we define

$$\nabla_1 \nabla_2 z(x_1, x_2, \ldots, x_n) = \nabla_1[z(x_1, x_2 \ldots, , x_n) - z(x_1, x_2 - 1, x_3 \ldots, x_n)].$$

and so on. Let B be a bounded domain in N_0^n defined by $B = \prod_{i=1}^n [0, c_i]$. For any real-valued function z defined on B we denote by $\sum_B z(x)$ the n-fold sum $\sum_{x_1=1}^{c_1} \cdots \sum_{x_n=1}^{c_n} z(x_1, \ldots, x_n)$ and for any $x \in B$ we denote by $\sum_{B_x} z(y)$ the n-fold sum $\sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} z(y_1, \ldots, y_n)$. We denote by $G(B)$ the class of functions $z : B \to R$ for which $\nabla_1 \ldots \nabla_n z(x)$ exist and such that

$$z(0, x_2, \ldots, x_n) = z(x_1, 0, x_2, \ldots, x_n) = \cdots = z(x_1, \ldots, x_{n-1}, 0) = 0. \tag{10}$$

We now state our result to be proved in this section.

Theorem 2. *Let $z \in G(B)$. Then*

$$\sum_B | z(x) || \nabla_1 \ldots \nabla_n z(x) | \tag{11}$$
$$\leq \Big(\sum_B (\prod_{i=1}^n x_i)(\sum_{B_x} | \nabla_1 \ldots \nabla_n z(y) |^2)\Big)^{\frac{1}{2}} \cdot \Big(\sum_B | \nabla_1 \ldots \nabla_n z(x) |^2\Big)^{\frac{1}{2}}.$$

Proof. For any $z \in G(B)$ we have the following identity

$$z(x) = \sum_{B_x} \nabla_1 \ldots \nabla_n z(y). \tag{12}$$

From (12) and by using the Schwarz inequality in summation form we observe that

$$| z(x) | \leq \sum_{B_x} | \nabla_1 \ldots \nabla_n z(y) | \leq \Big(\prod_{i=1}^{n} x_i\Big)^{\frac{1}{2}} \big(\sum_{B_x} | \nabla_1 \ldots \nabla_n z(y) |^2\big)^{\frac{1}{2}}. \quad (13)$$

Now by using Schwarz inequality in summation form we have

$$\sum_{B} | z(x) || \nabla_1 \ldots \nabla_n z(x) | \leq \big(\sum_{B} | z(x) |^2\big)^{\frac{1}{2}} \big(\sum_{B} | \nabla_1 \ldots \nabla_n z(x) |^2\big)^{\frac{1}{2}}. \quad (14)$$

Using (13) on the right side of (14) we get the required inequality in (11). This completes the proof of Theorem 2.

Remark 2. It is easy to observe that

$$\begin{aligned} \sum_{B} \big(\prod_{i=1}^{n} x_i\big) \big(\sum_{B_x} | \nabla_1 \ldots \nabla_n z(y) |^2\big) &\leq \Big(\sum_{B} \big(\prod_{i=1}^{n} x_i\big)\Big) \big(\sum_{B} | \nabla_1 \ldots \nabla_n z(y) |^2\big) \quad (15) \\ &= \big(\frac{1}{2^n} \prod_{i=1}^{n} c_i(c_i+1)\big) \big(\sum_{B} | \nabla_1 \ldots \nabla_n z(x) |^2\big). \end{aligned}$$

Now using (15) on the right side of (11) we have the following inequality

$$\sum_{B} | z(x) || \nabla_1 \ldots \nabla_n z(x) | \leq \big(\frac{1}{2^n} \prod_{i=1}^{n} c_i(c_i+1)\big)^{\frac{1}{2}} \sum_{B} | \nabla_1 \ldots \nabla_n z(x) |^2 . \quad (16)$$

If we take $n = 1$ in (16) and denote $c_1 = m, \nabla_1 z = \nabla z, x_1 = x$, then the inequality (16) reduces to the inequality

$$\sum_{x=0}^{m} | z(x) || \nabla z(x) | \leq \big(\frac{1}{2} m(m+1)\big)^{\frac{1}{2}} \sum_{x=0}^{m} | \nabla z(x) |^2 . \quad (17)$$

Here we note that the constant appearing in (17) is larger than the constant obtained in the discrete analogue of the Opial's inequality given by J.S.W. Wong in [7]. The main reason for it is the wastage at (15) in the general case of the proof of inequality (16).

In concluding we note that the inequalities established in Theorems 1 and 2 are different from the Opial type inequalities in two and many independent variables recently established by the authors in [3-6, 8]. We believe that the inequalities obtained in this paper are new to the literature.

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