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NONNULL DISTRIBUTION OF LRC FOR TESTING HOMOGENEITY OF COVARIANCE MATRICES OF COMPLETELY SYMMETRIC GAUSSIAN MODELS

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Abstract. The nonnull moments of the likelihood ratio statistic for testing equality of covariance matrices of completely symmetric Gaussian models are obtained in terms of the Lauricella's hypergeometric functions and also in terms of zonal polynomials. Then the nonnull asymptotic distribution of the statistic is derived under certain alternatives for unequal samples.

1.Introduction

Let $\underset{\sim}{X}_1, \ldots, \underset{\sim}{X}_m$ be random vectors of order $p \times 1$ which are distributed independently as multivariate normal with mean vectors $\underset{\sim}{\mu}_1, \ldots, \underset{\sim}{\mu}_m$ and positive definite covariance matrices $\sum_1, \ldots, \sum_m$ respectively. Further, let $\sum_i = \sigma_i^2[(1-\rho_i)I_p + \rho_i \underset{\sim}{e}\underset{\sim}{e}']$ and $\underset{\sim}{\mu}_i = \mu_i \underset{\sim}{e}$, where $\sigma_i^2(>0)$, ρ_i and μ_i are unknown scalars, $i = 1, \cdots, m$; Ip is the identity matrix of order p and $\underset{\sim}{e} = (1, \cdots, 1)'_{p\times 1}$. In the literature such a model is known as completely symmetric model.

Consider the null hypothesis

$$H_0 \ : \sum_1 = \ldots = \sum_m = \sigma^2[(1-\rho)I_p + \rho \underset{\sim}{e}\underset{\sim}{e}'] \tag{1.1}$$

against the general alternative.

Krishnaiah and Pathak (1967), Han (1975) and Gupta and Nagar (1986. 1987) considered the problem of testing homogeneity of covariance matrices under the intraclass correlation model and have derived the null and nonull distributions of the test statistic (also see Gupta et al. 1975). Tayal et al. (1989) have derived the distribution of the likelihood ratio test statistic under H_0.

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In this paper we consider the problem of nonnull distribution. The nonnull moments of the likelihood ratio test statistic are derived in section 2 as a product of two Lauricella's hypergeometric functions, and also in series involving zonal polynomials when the sample sizes are equal. In section 3 and 4, the asymptotic nonnull distribution of $-2\ln\Lambda$ has been derived for certain alternatives.

2. Likelihood ratio statistic and its moments.

Suppose independent random samples $\underset{\sim}{X}_{i1},\cdots,\underset{\sim}{X}_{iN_i}$ of size N_i are taken form the i^{th} population, $i=1,\cdots,m$. Let Q be a $p\times p$ orthogonal matrix with first row $\underset{\sim}{e}'/\sqrt{p}$, then $Q\sum_i Q' = diag(a_i,b_i,\cdots,b_i) = \sum_i^*$ and $\underset{\sim}{\nu}_i = Q\underset{\sim}{\mu}_i = \mu_i Q\underset{\sim}{e} = \sqrt{p}\mu_i\underset{\sim}{e}_1$, say, where $a_i=\sigma_i^2[1+(p-1)\rho_i]$, $b_i=\sigma_i^2(1-\rho_i)$ and $\underset{\sim}{e}_1=(1,0,\cdots,0)'_{p\times 1}$. The null hypothesis of testing homogeneity of covariance matrices is equivalent to $H_0: a_1=\ldots=a_m=a; b_1=\ldots=b_m=b$.

Let $\underset{\sim}{Y}_{iu}=Q\underset{\sim}{X}_{iu}$, $u=1,\cdots,N_i$, then $\underset{\sim}{Y}_{iu}\sim N(\underset{\sim}{\nu}_i,\sum_i^*)$. Further let, $W_{1i}=\sum_{u=1}^{N_i}(y_{i1u}-y_{i1\cdot})^2$, $W_{2i}=\sum_{j=2}^{p}\sum_{u=1}^{N_i}y_{iju}^2$, $W_1=\sum_{i=1}^m W_{1i}$,and $W_2=\sum_{i=1}^m W_{2i}$, where $y_{i1\cdot}=\frac{1}{N_i}\sum_{u=1}^{N_i}y_{i1u}$, y_{iju} being the j^{th} component of $\underset{\sim}{Y}_{iu}$.

The likelihood ratio statistic can easily be written as (see Tayal et al. 1987),

$$\Lambda = \Lambda_1\cdot\Lambda_2 \tag{2.1}$$

where

$$\Lambda_1 = \frac{N_0^{N_0/2}\prod_{i=1}^m W_{1i}^{N_i/2}}{\prod_{i=1}^m N_i^{N_i/2}(\sum_{i=1}^m W_{1i})^{N_0/2}} \tag{2.2}$$

$$\Lambda_2 = \frac{[N_0(p-1)]^{N_0(p-1)/2}\prod_{i=1}^m W_{2i}^{N_i(p-1)/2}}{\prod_{i=1}^m [N_i(p-1)]^{N_i(p-1)/2}\big(\sum_{i=1}^m W_{2i}\big)^{N_0(p-1)/2}}, \tag{2.3}$$

and $N_0=\sum_{i=1}^m N_i$. Further, Λ_1 and Λ_2 are independent. We now derive two results giving the nonnull moments of Λ_1 and Λ_2 respectively.

Lemma 2.1. *The h^{th} nonnull moment of Λ_1 defined in (2.2) is given by*

$$E(\Lambda_1^h) = \Big[\frac{N_0^{N_0/2}}{\prod_{i=1}^m N_i^{N_i/2}}\Big]^h\cdot\frac{\Gamma[(N_0-m)/2]}{\Gamma[\{N_0(1+h)-m\}/2]}$$

$$\cdot \prod_{i=1}^{m}\Big[\frac{\Gamma[N_i(1+h)/2-1/2](\eta_1 a_i^{-1})^{(N_i-1)/2}}{\Gamma[(N_i-1)/2]}\Big]$$
$$\cdot F_D^{(m)}\Big(\frac{1}{2}(N_0-m);\ \frac{1}{2}N_1(1+h)-\frac{1}{2},\cdots,\frac{1}{2}N_m(1+h)-\frac{1}{2};$$
$$\frac{1}{2}N_0(1+h)-\frac{1}{2}m;\ 1-\eta_1 a_1^{-1},\cdots,1-\eta_1 a_m^{-1}\Big) \tag{2.4}$$

where $|1-\eta_1 a_i^{-1}|<1$, $i=1,\cdots,m$ *and* $F_D^{(m)}$ *is the Lauricella's hypergeometric function (Slater, 1966).*

Proof. Since $W_{11}/a_1,\ldots,W_{1m}/a_m$ are independent chi-squares with $N_1-1,\ldots,N_m-1$ d.f. respectively, we get the h^{th} nonnull moment of Λ_1 as

$$E(\Lambda_1^h) = \Big[\frac{N_0^{N_0/2}}{\prod_{i=1}^{m} N_i^{N_i/2}}\Big]^h \int_{w_{11}>0}\cdots\int_{w_{1m}>0}\prod_{i=1}^{m}\Big[\frac{w_{1i}}{\sum_{i=1}^{m} w_{1i}}\Big]^{N_i h/2}$$
$$\cdot \prod_{i=1}^{m}\Big[\frac{w_{1i}^{\frac{1}{2}(N_i-1)-1}\exp\{-w_{1i}/2a_i\}}{(2a_i)^{(N_i-1)/2}\Gamma[(N_i-1)/2]}\Big]\, dw_{11}\ldots dw_{1m}. \tag{2.5}$$

Replacing $[\sum_{i=1}^{m} w_{1i}]^{-N_0h/2}$ by an equivalent gamma integral. namely

$$\Gamma(N_0h/2)2^{N_0h/2}\Big[\sum_{i=1}^{m} w_{1i}\Big]^{-N_0h/2} = \int_0^\infty exp\Big[-\Big(\sum_{i=1}^{m} w_{1i}\Big)x/2\Big]\cdot x^{N_0h/2-1}dx$$

for $Re(N_0h/2)>0$, changing the order of integration and integrating out $w_{11},\ldots,w_{1m}$, we have

$$E(\Lambda_1^h) = \Big[\frac{N_0^{N_0/2}}{\prod_{i=1}^{m} N_i^{N_i/2}}\Big]^h [\Gamma(N_0h/2)]^{-1}\prod_{i=1}^{m}\Big[\frac{\Gamma[N_i(1+h)/2-1/2]}{a_i^{(N_i-1)/2}\Gamma[(N_i-1)/2]}\Big]$$
$$\cdot \int_0^\infty x^{N_0h/2-1}\prod_{i=1}^{m}(x+a_i^{-1})^{-[N_i(1+h)/2-1/2]}dx. \tag{2.6}$$

Substituting $u=1/(1+\eta_1 x)$ in the above integrand and using (2.1) of Gupta and Nagar (1987), we can easily get (2.4).

Lemma 2.2. *The* h^{th} *nonnull moment of* Λ_2 *defined in* (2.3) *is given by*

$$E(\Lambda_2^h) = \Big[\frac{N_0^{N_0/2}}{\prod_{i=1}^{m} N_i^{N_i/2}}\Big]^{h(p-1)}\frac{\Gamma[N_0(p-1)/2]}{\Gamma[N_0(p-1)(1+h)/2]}\prod_{i=1}^{m}\Big[\frac{\Gamma[N_i(p-1)(1+h)/2](\eta_2 b_i^{-1})^{N_i(p-1)/2}}{\Gamma[N_i(p-1)/2]}\Big]$$
$$\cdot F_D^{(m)}\Big(\frac{1}{2}N_0(p-1);\ \frac{1}{2}N_1(p-1)(1+h),\cdots,\frac{1}{2}N_m(p-1)(1+h);$$
$$\frac{1}{2}N_0(p-1)(1+h);\ 1-\eta_2 b_1^{-1},\cdots,1-\eta_2 b_m^{-1}\Big) \tag{2.7}$$

where $|1-\eta_2 b_i^{-1}|<1,\ i=1,\cdots,m.$

Proof. In this case $W_{21}/b_1,\ldots,W_{2m}/b_m$ are independent chi-squared variates with $N_1(p-1),\ldots,N_m(p-1)$ d.f. respectively. Therefore the h^{th} moment of Λ_2 can be obtained as in Lemma 2.1.

From Lemmas 2.1 and 2.2 and independence of Λ_1 and Λ_2 we obtain the following result.

Theorem 2.1. *The h^{th} nonnull moment of the test statistic Λ defined by (2.1) is given by*

$$\begin{aligned} E(\Lambda^h) &= \frac{N_0^{N_0ph/2}}{\prod_{i=1}^m N_i^{N_iph/2}} \cdot \frac{\Gamma[(N_0-m)/2]\Gamma[N_0(p-1)/2]}{\Gamma[\{N_0(1+h)-m\}/2]\Gamma[N_0(p-1)(1+h)/2]} \\ &\cdot \prod_{i=1}^m \Big[\frac{\Gamma[N_i(1+h)/2-1/2](\eta_1 a_i^{-1})^{(N_i-1)/2}}{\Gamma[(N_i-1)/2]} \frac{\Gamma[N_i(p-1)(1+h)/2](\eta_2 b_i^{-1})^{N_i(p-1)/2}}{\Gamma[N_i(p-1)/2]}\Big] \\ &\cdot F_D^{(m)}(\frac{1}{2}(N_0-m); \frac{1}{2}N_1(1+h)-\frac{1}{2},\cdots,\frac{1}{2}N_m(1+h)-\frac{1}{2}; \\ &\quad \frac{1}{2}N_0(1+h)-\frac{m}{2}; 1-\eta_1 a_1^{-1},\cdots,1-\eta_1 a_m^{-1}) \\ &\cdot F_D^{(m)}(\frac{1}{2}N_0(p-1); \frac{1}{2}N_1(p-1)(1+h),\cdots,\frac{1}{2}N_m(p-1)(1+h); \\ &\quad \frac{1}{2}N_0(p-1)(1+h); 1-\eta_2 b_1^{-1},\cdots,1-\eta_2 b_m^{-1}) \end{aligned} \tag{2.8}$$

where $|1-\eta_1 a_i^{-1}|<1$ *and* $|1-\eta_2 b_i^{-1}|<1,\ i=1,\cdots,m.$

When $N_i=N,\ i=1,\cdots,m$, the h^{th} nonnull moment of the test statistic Λ can be written in terms of zonal polynomials by substituting equivalent representation of Lauricella's hypergeometric function (see Gupta and Nagar, 1988a; eqn. 2.4) as

$$\begin{aligned} E(\Lambda^h) &= \frac{m^{mNph/2}\Gamma^m[\frac{N}{2}(1+h)-\frac{1}{2}]\Gamma^m[\frac{N(p-1)(1+h)}{2}]}{\Gamma[\frac{N-1}{2}]\Gamma^m[\frac{N(p-1)}{2}]} \\ &\cdot |\eta_1\Delta_1^{-1}|^{(N-1)/2}|\eta_2\Delta_2^{-1}|^{N(p-1)/2} \\ &\cdot \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\mathcal{K}}\sum_{J}(\frac{1}{2}N(1+h)-\frac{1}{2})_{\mathcal{K}}\Big(\frac{N(p-1)(1+h)}{2}\Big)_J \\ &\cdot \frac{C_{\mathcal{K}}(I_m-\eta_1\Delta_1^{-1})}{k!} \cdot \frac{C_j(I_m-\eta_2\Delta_2^{-1})}{j!} \\ &\cdot \frac{\Gamma[\frac{m}{2}(N-1)+k]\Gamma[\frac{mN(p-1)}{2}+j]}{\Gamma[\frac{mN(1+h)}{2}-\frac{m}{2}+k]\Gamma[\frac{mN(p-1)(1+h)}{2}+j]} \end{aligned} \tag{2.9}$$

where $\Delta_1 = diag(a_1, \ldots, a_m)$; $\Delta_2 = diag(b_1, \ldots, b_m)$; $||I_m - \eta_i \Delta_i^{-1}|| < 1$; $i = 1, 2$, and $C_K(S)$ is a zonal polynomial (see James, 1964) of order k in the latent roots of the $m \times m$ symmetric matrix S; $\kappa = (k_1, \ldots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, $k_1 + \ldots + k_m = k$.

3. Asymptotic nonnull distribution for unequal samples

In this section we derive the asymptotic nonnull distribution of a mutiple of $-2\ln\Lambda$ when the sample sizes are unequal. Let $d_i = \frac{N_i}{N_0}$, $i = 1, \cdots, m$,

$$\delta_0 = \frac{1}{12(m-1)}\Big[\frac{(11p-9)}{2(p-1)}(\sum_{i=1}^m d_i^{-1} - 1) - \frac{3}{2}(m-1)(m+3)\Big], \quad N_0 = M_0 + 2\delta_0.$$

$N_0\tau = M_0$, and consider the alternative hypothesis

$$A_{M_0} \;:\; 1 - \eta_1 a_i^{-1} = \frac{2p_i}{d_i M_0}; 1 - \eta_2 b_i^{-1} = \frac{2q_i}{d_i M_0(p-1)}, i = 1, \cdots, m. \tag{3.1}$$

Under A_{M_0}, the characteristic function of $W = -2\tau\ln\Lambda$ is derived from Theorem 2.1 as

$$\begin{aligned}\varphi_W(t) = \varphi_{H_0}(t)\prod_{i=1}^m\Big\{(1 - \frac{2p_i}{d_i M_0})^{\frac{1}{2}M_0 d_i + d_i\delta_0 - \frac{1}{2}}(1 - \frac{2q_i}{d_i M_0(p-1)})^{(d_i\delta_0 + \frac{1}{2}M_0 d_i)(p-1)}\Big\}\\ \cdot F_D^{(m)}(\frac{1}{2}M_0 + \delta_0 - \frac{m}{2}; \frac{1}{2}M_0 d_1\theta + d_1\delta_0 - \frac{1}{2}, \cdots, \frac{1}{2}M_0 d_m\theta + d_m\delta_0 - \frac{1}{2};\\ \frac{1}{2}M_0\theta + \delta_0 - \frac{m}{2}; \frac{2p_1}{d_1 M_0}, \cdots, \frac{2p_m}{d_m M_0})\\ \cdot F_D^{(m)}((\frac{1}{2}M_0 + \delta_0)(p-1); (\frac{1}{2}M_0 d_1\theta + d_1\delta_0)(p-1)(\frac{1}{2}M_0 d_m\theta + d_m\delta_0)(p-1);\\ (\frac{1}{2}M_0\theta + \delta_0)(p-1); \frac{2q_1}{d_1 M_0(p-1)}, \cdots, \frac{2q_m}{d_m M_0(p-1)})\end{aligned} \tag{3.2}$$

where $\theta = 1 - 2\omega t$, $\omega = \sqrt{-1}$ and $\varphi_{H_0}(t)$ is the characteristic function of $-2\tau\ln\Lambda$ when the null hypothesis is true and can be obtained from (3.1) of Tayal et al. (1989) by putting $h = -2\omega\tau t$. The asymptotic expansion of $\varphi_{H_0}(t)$ is given below:

Lemma 3.1. *Let $N_i = d_i N_0$, $d_i > 0$, $i = 1, \cdots, m$ with $\sum_{i=1}^m d_i = 1$. When H_0 is true the characteristic function of $W = -2\tau\ln\Lambda$, can be expanded as*

$$\varphi_{H_0}(t) = \theta^{-f/2}[1 + \frac{\gamma_2}{M_0^2}\{\theta^{-2} - 1\} + O(M_0^{-3})] \tag{3.3}$$

where $f = 2(m-1)$, and

$$\gamma_2 = \frac{1}{2}[\sum_{i=1}^m d_i^{-2} - \frac{m(m+1)(m+2)}{6} - 2f\delta_0^2]. \tag{3.4}$$

Proof. See Anderson (1984, p. 311).

The asymptotic expansions of the Lauricella's hypergeometric functions involved in (3.2) are also given in the following two results:

Lemma 3.2. *For $N_i = d_i N_0$, $d_i > 0$, $i = 1, \cdots, m$ with $\sum_{i=1}^{m} d_i = 1$, the asymptotic expansion of $F_D^{(m)}$ involved in the characteristic function of $-2\tau \ln \Lambda$ is*

$$\begin{aligned} &F_D^{(m)}(\frac{1}{2}M_0 + \delta_0 - \frac{m}{2}; \frac{1}{2}M_0 d_1\theta + d_1\delta_0 - \frac{1}{2}, \cdots, \frac{1}{2}M_0 d_m\theta \\ &+ d_m\delta_0 - \frac{1}{2}; \frac{1}{2}M_0\theta + \delta_0 - \frac{m}{2}; \frac{2p_1}{d_1 M_0}, \cdots, \frac{2p_m}{d_m M_0}) \\ &= \exp(\sum_{i=1}^{m} p_i)[1 + \frac{1}{M_0}\{A_0 + A_1\theta^{-1}\} + O(M_0^{-2})] \end{aligned} \tag{3.5}$$

where

$$A_0 = (2\delta_0 - m)(\sum_{i=1}^{m} p_i) + (\sum_{i=1}^{m} p_i)^2 \tag{3.6}$$

and

$$A_1 = m(\sum_{i=1}^{m} p_i) - (\sum_{i=1}^{m} p_i)^2 + \sum_{i=1}^{m}(p_i^2/d_i) - \sum_{i=1}^{m}(p_i/d_i). \tag{3.7}$$

Proof. Expanding $F_D^{(m)}$ in the series form using (2.2) of Gupta and Nagar (1987). one has

$$\begin{aligned} &F_D^{(m)}(\frac{1}{2}M_0 + \delta_0 - \frac{m}{2}; \frac{1}{2}M_0 d_1\theta + d_1\delta_0 - \frac{1}{2}, \cdots; \\ &\frac{1}{2}M_0 d_m\theta + d_m\delta_0 - \frac{1}{2}; \frac{1}{2}M_0\theta + \delta_0 - \frac{m}{2}; \frac{2p_1}{d_1 M_0}, \cdots, \frac{2p_m}{d_m M_0}) \\ &= \sum_{r=0}^{\infty}\sum_{R} G_1(R,t)\prod_{i=1}^{m}\Big\{(\frac{2p_i}{d_i M})^{r_i}/r_i!\Big\} \end{aligned} \tag{3.8}$$

where $R = (r_1, \ldots, r_m), r_1 + \ldots + r_m = r$, $\sum_R$ denotes summation over all such partitions, and

$$G_1(R,t) = \prod_{i=1}^{m}\Big\{(\frac{1}{2}M_0 d_i\theta + d_i\delta_0 - \frac{1}{2})_{r_i}\Big\}\frac{(\frac{1}{2}(M_0 - m) + \delta_0)_r}{(\frac{1}{2}(M_0\theta - m) + \delta_0)_r}.$$

Expanding logarithm of $G_1(R,t)$, and converting back we get

$$\begin{aligned} G_1(R,t) &= \theta^{-r}[\prod_{i=1}^{m}(M_0 d_i\theta/2)^{r_i}]\Big\{1 + \sum_{i=1}^{m}\frac{r_i(2d_i\delta_0 + r_i - 2)}{M_0 d_i\theta} + O(M_0^{-2})\Big\} \\ &\cdot\Big\{1 + \frac{r(2\delta_0 - m + r - 1)}{M_0}(1 - \theta^{-1}) + O(M_0^{-2})\Big\}. \end{aligned} \tag{3.9}$$

Substituting G_1 (R,t) from (3.9) in (3.8), using multinomial summation formulae and summing over r one can easily get (3.5).

Lemma 3.3. *For $N_i = d_i N_0$, $d_i > 0$, $i = 1, \cdots, m$ with $\sum_{i=1}^{m} d_i = 1$, the asymptotic expansion of $F_D^{(m)}$ involved in the characteristic function of $-2\tau \ln \Lambda$ is*

$$\begin{aligned} &F_D^{(m)}((\frac{1}{2}M_0 + \delta_0)(p-1); (\frac{1}{2}M_0 d_1 \theta + d_1\delta_0)(p-1), \cdots, (\frac{1}{2}M_0 d_m \theta + d_m \delta_0)(p-1);\\ &\quad (\frac{1}{2}M_0\theta + \delta_0)(p-1); \frac{2q_1}{d_1 M_0 (p-1)}, \cdots, \frac{2q_m}{d_m M_0(p-1)})\\ &= \exp\Big(\sum_{i=1}^{m} q_i\Big) \cdot \Big[1 + \frac{1}{M_0}\{A_0' + A_1'\theta^{-1}\} + O(M_0^{-2})\Big] \end{aligned} \tag{3.10}$$

where

$$A_0' = 2\delta_0\Big(\sum_{i=1}^{m} q_i\Big) + \frac{1}{(p-1)}\Big(\sum_{i=1}^{m} q_i\Big)^2 \tag{3.11}$$

and

$$A_1' = \frac{1}{(p-1)}\Big[\sum_{i=1}^{m}(q_i^2/d_i) - (\sum_{i=1}^{m} q_i)^2\Big]. \tag{3.12}$$

Proof. Similar to the proof of Lemma 3.2.

Lemma 3.4. *For large M_0, we have*

$$\begin{aligned} &\prod_{i=1}^{m}\Big\{(1 - \frac{2p_i}{d_i M_0})^{\frac{1}{2}M_0 d_i + d_i\delta_0 - \frac{1}{2}}(1 - \frac{2q_i}{d_i M_0(p-1)})^{(\frac{1}{2}M_0 d_i + d_i\delta_0)(p-1)}\Big\}\\ &= \exp[-\sum_{i=1}^{m}(p_i + q_i)] \cdot \Big[1 - \frac{1}{M_0}\Big\{\frac{1}{(p-1)}\sum_{i=1}^{m} q_i\{2\delta_0(p-1) + \frac{q_i}{d_i}\}\\ &\quad + \sum_{i=1}^{m} p_i\{2\delta_0 + \frac{1}{d_i}(p_i - 1)\}\Big\} + O(M_0^{-2})\Big]. \end{aligned} \tag{3.13}$$

Proof. Expanding the logarithm of

$$\prod_{i=1}^{m}\Big\{(1 - \frac{2p_i}{d_i M_0})^{\frac{1}{2}M_0 d_i + d_i\delta_0 - \frac{1}{2}}(1 - \frac{2q_i}{d_i M_0(p-1)})^{(\frac{1}{2}M_0 d_i + d_i\delta_0)(p-1)}\Big\}$$

and converting back, one can easily get (3.14).

Substituting (3.3), (3.5), (3.10) and (3.13) in (3.2) and inverting the resulting expansion of the characteristic function, one gets the following result:

Theorem 3.1. *The asymptotic expansion of the distribution of* $W = -2\tau \ln \Lambda$ *under the alternative hypothesis stated in* (3.1) *is give by*

$$P[W \le w] = P[\chi_f^2 \le w] + \frac{c}{M_0}\{P[\chi_{f+2}^2 \le w] - P[\chi_f^2 \le w]\} + O(M_0^{-2}) \qquad (3.14)$$

where

$$C = -\Big[(\sum_{i=1}^{m} \frac{p_i}{d_i}) - m(\sum_{i=1}^{m} p_i) + (\sum_{i=1}^{m} p_i)^2 - (\sum_{i=1}^{m} \frac{p_i^2}{d_i}) + \frac{1}{p-1}\Big((\sum_{i=1}^{m} q_i)^2 - \sum_{i=1}^{m}(\frac{q_i^2}{d_i})\Big)\Big] \qquad (3.15)$$

and χ_v^2 *is the chi-square variate with* v *d.f.*

4. Asymptotic nonnull distribution for equal samples

In this section we derive the asymptotic nonnull distribution of $-2\ln\Lambda$ when the sample sizes are equal. Let $W = -(2M/N)\ln\Lambda$ where $M = N - 2\delta$ (δ is a constant to be specified later). Then the characteristic function $\varphi(t)$ of W is

$$\varphi(t) = \varphi_1(t).\varphi_2(t) \qquad (4.1)$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are the characteristic functions of $-(\frac{2M}{N})\ln\Lambda_1$ and $-(\frac{2M}{N})\ln\Lambda_2$ respectively. In this section we derive the asymptotic distribution of W by using the asymptotic expansion of $\varphi(t)$ as $M \to \infty$, for the following sequence of alternatives (see Khatri and Srivastava, 1974; Gupta and Nagar, 1987)

$$\text{(i)} \quad I - \eta_1\Delta_1^{-1} = \frac{2P_1}{M}; \qquad I - \eta_2\Delta_2^{-1} = \frac{2P_2}{(p-1)M}$$

$$\text{(ii)} \quad I - \eta_1^{-1}\Delta_1 = \frac{2Q_1}{M}; \qquad I - \eta_2\Delta_2^{-1} = \frac{2P_2}{(p-1)M}$$

$$\text{(iii)} \quad I - \eta_1^{-1}\Delta_1 = \frac{2Q_1}{M}; \qquad I - \eta_2^{-1}\Delta_2 = \frac{2Q_2}{(p-1)M}$$

$$\text{(iv)} \quad I - \eta_1\Delta_1^{-1} = \frac{2P_1}{M}; \qquad I - \eta_2^{-1}\Delta_2 = \frac{2Q_2}{(p-1)M}$$

where P_1, P_2, Q_1 and Q_2 are fixed matrices.

Case (i):

$$I - \eta_1\Delta_1^{-1} = \frac{2P_1}{M}; \; I - \eta_2\Delta_2^{-1} = \frac{2P_2}{(p-1)M}$$

Following the approximation procedure described in Gupta and Nagar (1987) (also see Khatri and Srivastava, 1974; Gupta and Nagar, 1988a 1988b) we obtain the asymptotic expansions for the characteristic functions $\varphi_j(t)$, $j = 1, 2$ as follows:

$$\varphi_j(t) = \theta^{-(m-1)/2}[1 + \frac{A^{(j)}}{M}(\theta^{-1} - 1) + \frac{1}{M^2}(B_0^{(j)} + B_1^{(j)}\theta^{-1} + B_2^{(j)}\theta^{-2}) + O(M^3), \qquad j = 1, 2 \qquad (4.2)$$

where

$$
\begin{aligned}
A^{(1)} &= -(\delta-\frac{1}{2})(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_1^2-\frac{(\operatorname{tr}P_1)^2}{m}\\
B_0^{(1)} &= \frac{1}{2}[-(\delta-\frac{1}{2})(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_1^2-\frac{(\operatorname{tr}P_1)^2}{m}]^2+r_1\\
&\quad +\frac{4}{3}[\frac{(\operatorname{tr}P_1)^3}{m^2}-\operatorname{tr}P_1^3]+2(\delta-\frac{1}{2})[\frac{(\operatorname{tr}P_1)^2}{m}-\operatorname{tr}P_1^2]\\
B_1^{(1)} &= -\Big[\{-(\delta-\frac{1}{2})(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_1^2-\frac{(\operatorname{tr}P_1)^2}{m}\}^2\\
&\quad +4(\delta-\frac{1}{2})\{\frac{(\operatorname{tr}P_1)^2}{m}-\operatorname{tr}P_1^2\}+\frac{4(\operatorname{tr}P_1)}{m}[\frac{(\operatorname{tr}P_1)^2}{m}-\operatorname{tr}P_1^2]\\
&\quad +\frac{2}{m}[\frac{(\operatorname{tr}P_1)^2}{m}-\operatorname{tr}P_1^2]\Big]\\
B_2^{(1)} &= -(B_1^{(1)}+B_0^{(1)})\\
r_1 &= \frac{2}{3m^2}[-\frac{3}{2}m^2(\delta-\frac{1}{2})^2(m-1)+\frac{1}{2}m(\delta-\frac{1}{2})(m^2-1)]\\
A^{(2)} &= -\delta(p-1)(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_2^2-\frac{(\operatorname{tr}P_2)^2}{m}\\
B_0^{(2)} &= \frac{1}{2}[-\delta(p-1)(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_2^2-\frac{(\operatorname{tr}P_2)^2}{m}]^2+r_2\\
&\quad +\frac{4}{3}[\frac{(\operatorname{tr}P_2)^3}{m^2}-\operatorname{tr}P_2^3]+2\delta(p-1)[\frac{(\operatorname{tr}P_2)^2}{m}-\operatorname{tr}P_2^2]\\
B_1^{(2)} &= -\Big[\{-\delta(p-1)(m-1)+\frac{m^2-1}{6m}+\operatorname{tr}P_2^2-\frac{(\operatorname{tr}P_2)^2}{m}\}^2\\
&\quad +4\delta(p-1)[\frac{(\operatorname{tr}P_2)^2}{m}-\operatorname{tr}P_2^2]+\frac{4(\operatorname{tr}P_2)}{m}\{\frac{(\operatorname{tr}P_2)^2}{m}-\operatorname{tr}P_2^2\}\\
&\quad +\frac{2}{m}\{\frac{(\operatorname{tr}P_2)^2}{m}-\operatorname{tr}P_2^2\}\Big]\\
B_2^{(2)} &= -(B_1^{(2)}+B_0^{(2)})\\
r_2 &= \frac{2}{3m^2}[-\frac{3}{2}m^2\delta^2(p-1)^2(m-1)+\frac{1}{2}m\delta(p-1)(m^2-1)].
\end{aligned}
$$

Multiplying $\varphi_1(t)$ and $\varphi_2(t)$, we get the characteristic function $\varphi(t)$ as

$$
\varphi(t)=\theta^{-(m-1)}[1-\frac{C}{M}(\theta^{-1}-1)+\frac{1}{M^2}\{T_0+T_1\theta^{-1}+T_2\theta^{-2}\}+O(M^{-3})] \tag{4.3}
$$

where

$$
\begin{aligned}
C &= -[-2\delta(m-1)+\frac{1}{2}(m-1)+\frac{(m^2-1)p}{6m(p-1)}+(\operatorname{tr}P_1^2)-\frac{(\operatorname{tr}P_1)^2}{m}\\
&\quad +\frac{1}{p-1}((\operatorname{tr}P_2^2)-\frac{(\operatorname{tr}P_2)^2}{m})].
\end{aligned}
$$

Now choose δ such that $-2\delta(m-1)+\frac{1}{2}(m-1)+\frac{(m^2-1)p}{6m(p-1)}=0$, giving $\delta=\frac{m(4p-3)+p}{12m(p-1)}$. For this value of δ, C, T_0, T_1 and T_2 are given by

$$
\begin{aligned}
C &= C_1 + \frac{C_1'}{p-1} \\
T_0 &= \frac{1}{2}C^2 + \frac{4}{3}C_2 + 2(\delta - \frac{1}{2})C_1 + \frac{4}{3(p-1)^2}C_2' + \frac{2\delta}{(p-1)}C_1' + r \\
T_1 &= -\Big\{C^2 + 4(\delta - \frac{1}{2})C_1 + \frac{4(\operatorname{tr} P_1)}{m}C_1 + \frac{2}{m}C_1 + \frac{4\delta}{(p-1)}C_1' \\
&\quad + \frac{4(\operatorname{tr} P_2)}{m(p-1)^2}C_1' + \frac{2}{m(p-1)^2}C_1'\Big\} \\
T_2 &= -(T_0 + T_1)
\end{aligned}
$$

where $C_1=\frac{1}{m}(\operatorname{tr} P_1)^2-(\operatorname{tr}P_1^2)$, $C_1'=\frac{1}{m}(\operatorname{tr}P_2)^2-(\operatorname{tr}P_2^2)$, $C_2=\frac{1}{m^2}(\operatorname{tr}P_1)^3-(\operatorname{tr}P_1^3)$, $C_2'=\frac{1}{m^2}(\operatorname{tr}P_2)^3-(\operatorname{tr}P_2^3)$, and $r=2\delta^2(m-1)-\frac{(m-1)(4m+1)}{6m}$.

Inverting (4.3), we get the asymptotic distribution of W as

$$
\begin{aligned}
P[W \le w] &= P[\chi_f^2 \le w] + \frac{C}{M}\{P[\chi_f^2 \le w] - P[\chi_{f+2}^2 \le w]\} \qquad (4.4)\\
&\quad + \frac{1}{M^2}\{T_0 P[\chi_f^2 \le w] + T_1 P[\chi_{f+2}^2 \le w] + T_2[\chi_{f+4}^2 \le w]\} + O(M^{-3})
\end{aligned}
$$

where $f=2(m-1)$

Case (ii):

$$I-\eta_1^{-1}\Delta_1=\frac{2Q_1}{M};\ I-\eta_2\Delta_2^{-1}=\frac{2P_2}{(p-1)M}.$$

Replacing P_1 by $-Q_1(I+\frac{2Q_1}{M})$ in the coefficients in (4.4) and rearranging certain terms, various coefficients for this case are

$$C = C_1^* + \frac{1}{(p-1)}C_1'$$

$$
\begin{aligned}
T_0 &= \frac{1}{2}C^2 + 2(\delta - \frac{1}{2})C_1^* + \frac{8}{3}C_2^* - \frac{4}{m}(\operatorname{tr}Q_1)C_1^* \\
&\quad + \frac{4}{3(p-1)^2}C_2' + \frac{2\delta}{(p-1)}C_1' + r \\
T_1 &= -\{C^2 + 4(\delta - \frac{1}{2})C_1^* - \frac{8}{m}(\operatorname{tr} Q_1)C_1^* + \frac{2}{m}C_1^* \\
&\quad + 4C_2^* + \frac{4\delta}{p-1}C_1' + \frac{4(\operatorname{tr} P_2)}{m(p-1)^2}C_1' + \frac{2}{m(p-1)^2}C_1'\} \\
T_2 &= -(T_0 + T_1),
\end{aligned}
$$

where

$$C_1^* = \frac{1}{m}(\operatorname{tr}Q_1)^2 - \operatorname{tr}Q_1^2, \qquad C_1' = \frac{1}{m}(\operatorname{tr}P_2)^2 - \operatorname{tr}P_2^2,$$

$$C_2^* = \frac{1}{m^2}(\text{tr}Q_1)^3 - \text{tr}Q_1^3 \quad \text{and} \quad C_2' = \frac{1}{m^2}(\text{tr}P_2)^3 - \text{tr}P_2^3.$$

The asymptotic distribution in this case is given by

$$\begin{aligned} P[W \le w] &= P[\chi_f^2 \le w] + \frac{C}{M}\{P[\chi_f^2 \le w] - P[\chi_{f+2}^2 \le w]\} \qquad (4.5) \\ &+ \frac{1}{M^2}\{T_0 P[\chi_f^2 \le w] + T_1 P[\chi_{f+2}^2 \le w] + T_2 P[\chi_{f+4}^2 \le w]\} + O(M^{-3}). \end{aligned}$$

Case (iii):

$$I - \eta_1^{-1}\Delta_1 = \frac{2Q_1}{M}; \; I - \eta_2^{-1}\Delta_2 = \frac{2Q_2}{(p-1)M}.$$

The asymptotic distribution is given by (4.5) by replacing P_2 by $-Q_2(I + \frac{2Q}{(p-1)M})$. Now the coefficients for this case are given by

$$C = C_1^* \; + \; \frac{1}{(p-1)}C_1^{*\prime}$$

$$\begin{aligned} T_0 &= \frac{1}{2}\Big[C_1^* + \frac{1}{(p-1)}C_1^{*\prime}\Big]^2 + 2(\delta - \frac{1}{2})C_1^* - \frac{4}{m}(\text{tr } Q_1)C_1^* + \frac{8}{3}C_2^* \\ &+ \frac{2\delta}{p-1}C_1^{*\prime} - \frac{4(\text{tr } Q_2)}{m(p-1)^2}C_1^{*\prime} + \frac{8}{3(p-1)^2}C_2^{*\prime} + r \\ T_1 &= -\Big\{\Big[C_1^* + \frac{1}{(p-1)}C_1^{*\prime}\Big]^2 + 4(\delta - \frac{1}{2})C_1^* - \frac{8}{m}(\text{tr } Q_1)C_1^* + \frac{2}{m}C_1^* + 4C_2^* \\ &+ \frac{4\delta}{(p-1)}C_1^{*\prime} - \frac{8(\text{tr } Q_2)}{m(p-1)^2}C_1^{*\prime} + \frac{2}{m(p-1)^2}C_1^{*\prime} + \frac{4}{(p-1)^2}C_2^{*\prime}\Big\} \\ T_2 &= -(T_0 + T_1) \end{aligned}$$

where $C_1^* = \frac{1}{m}(\text{tr } Q_1)^2 - \text{tr } Q_1^2$, $C_1^{*\prime} = \frac{1}{m}(\text{tr } Q_2)^2 - \text{tr } Q_2^2$,

$$C_2^* = \frac{1}{m^2}(\text{tr } Q_1)^3 - \text{tr } Q_1^3, \quad \text{and} \quad C_2^{*\prime} = \frac{1}{m^2}(\text{tr } Q_2)^3 - \text{tr } Q_2^3.$$

Case (iv):

$$I - \eta_1\Delta_1^{-1} = \frac{2P_1}{M}; \; I - \eta_2^{-1}\Delta_2 = \frac{2Q_2}{(p-1)M}.$$

The asymptotic distribution is given by (4.4) by replacing P_2 by $-Q_2(I + \frac{2}{M(p-1)}Q_2)$ where now,

$$C = C_1 \; + \; \frac{1}{(p-1)}C_1^{*\prime}$$

$$
\begin{aligned}
T_0 &= \frac{1}{2}\Big[C_1 + \frac{1}{(p-1)}C_1^{*\prime}\Big]^2 + \frac{4}{3}C_2 + 2(\delta - \frac{1}{2})C_1 + r + \frac{2\delta}{(p-1)}C_1^{*\prime} \\
&\quad - \frac{4}{m(p-1)^2}(\mathrm{tr}Q_2)C_1^{*\prime} + \frac{8}{3(p-1)^2}C_2^{*\prime} \\
T_1 &= -\Big\{\Big[C_1 + \frac{1}{(p-1)}C_1^{*\prime}\Big]^2 + 4(\delta - \frac{1}{2})C_1 + \frac{4(\mathrm{tr}P_1)}{m}C_1 + \frac{2}{m}C_1 + \frac{4\delta}{(p-1)}C_1^{*\prime} \\
&\quad - \frac{8(\mathrm{tr}Q_2)}{m(p-1)^2}C_1^{*\prime} + \frac{2}{m(p-1)2}C_1^{*\prime} + \frac{4}{(p-1)^2}C_2^{*\prime}\Big\}
\end{aligned}
$$

$T_2 = -(T_0 + T_1)$ and C_1, $C_1^{*\prime}$, C_2, $C_2^{*\prime}$ and r are defined above.

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