TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 1, Spring 1991

ON THE $||T|| \cdot C_1$ SUMMABILITY OF A SEQUENCE OF FOURIER COEFFICIENTS

M.L. MITTAL, G.PRASAD AND RAJESH KUMAR

Abstract. Mohanty and Nanda (1959) were the first to establish a result for the $(C,1)$ i.e. C_1-summability of the sequence $\{n\ B_n(x)\}$. Varshney (1959) improved the result for $(N, \frac{1}{n+1})\cdot C_1$ summability which was generalised by several investigators such as Sharma (1970), Singh (1963), Lal (1971), Khare and Singh (1988) etc. In this note, we have discussed $||T|| \cdot C_1$-summability of the sequence $\{n\ B_x(x)\}$ which includes the result due to Khare and Singh (1988).

1. Let Σu_n be a given infinite series with the sequence of partial sum $\{s_n\}$. Let $||T|| \equiv (a_{n,k})$ be infinite triangular matrix with real constants. Then sequence-to-sequence transformation.

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k, \qquad n = 0, 1, 2, \cdots;$$

defines the T-transform of the sequence $\{s_n\}$. Recall that the matrix elements $a_{n,k} = 0$ for each $k > n$, then the matrix is called *triangular*. The series Σu_n is said to be *T-summable* to s, if $\lim_{n\to\infty} t_n = s$

The regularity conditions for T-method are :

(1) There exists a constant $K : \Sigma_{k=0}^{n} \mid a_{n,k} \mid < K$, for each n;

(2) For every k, $\lim_{n\to\infty} a_{n,k} = 0$; and

(3) $\lim_{n\to\infty} \Sigma_0^n a_{n,k} = 1$.

The matrix T-reduces to Nörlund matrix generated by the sequence of coefficients $\{p_n\}$ if

$$a_{n,k} = \begin{cases} p_{n-k}/P_n, & \text{if } k \le n; \\ 0, & \text{if } k > n; \end{cases}$$

where $P_n = \sum_{r=0}^{n} p_r \neq 0$.

Received Oct. 24, 1989.

If the method of summability $||T||$ is applied to Cesaro means of order one, another method of summability $||T|| \cdot C_1$ is obtained.

2. Let $f(x)$ be a periodic function with period 2π and integrable in the sense of Lebesgue over an interval $(-\pi, \pi)$. Let the Fourier-series of $f(x)$ be

$$\frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^\infty A_n(x), \tag{2.1}$$

and the series conjugate to (2.1) is

$$\sum_{n=1}^\infty (b_n \cos nx - a_n \sin n\pi) \equiv \sum_{n=1}^\infty B_n(x). \tag{2.2}$$

We write

$$\begin{aligned} \phi(t) &= f(x+t) - f(x-t); \\ \Psi(t) &= f(x+t) - f(x-t) - \ell; \\ \Psi_1(t) &= \int_0^t |\psi(x)|\, du; \\ A_{n,r} &= \sum_{k=r}^n a_{n,k}; \end{aligned}$$

where ℓ is a constant, and $\mathcal{T} = [1/t]$ the integral part of $1/t$.

3. Mohanty and Nanda (1954) proved the following theorem :

Theorem A. *If*

$$\Psi(t) = o(1/\log(1/t)) \text{ as } t \to 0, \tag{3.1}$$

$$\text{and}$$

$$a_n = O(n^{-\delta});\ b_n = O(n^{-\delta}),\ O < \delta < 1,$$

then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value ℓ/π.

From this result they have deduce a well known criterion, the Hardy and Littlewood's test for the convergence of the conjugate series (2.2). Varshney (1959) improved Theorem A in the follwing form :

Theorem B. *If*

$$\int_0^t |\Psi(u)|\, du = o(\frac{t}{\log 1/t}), \text{ as } t \to 0, \tag{3.2}$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N, 1/n+1)$. C_1 *to the value* ℓ/π.

Result of Varshney was generalised by several workers for $(N,p_n)\cdot C_1$ summability of the sequence $\{nB_n(x)\}$ such as Sharma (1970), Lal (1971) using monotonocity on $\{p_n\}$. Dropping the monotonocity, very recently Khare and Singh (1988) proved $(N,p_n)\cdot C_1$ summability of the sequence $\{nB_n(x)\}$. They proved:

Theorem C. *Let* (N,p_n) *be a reqular Nörlund method defined by a sequence* $\{p_n\}$ *of complex numbers such that*

$$\sum_{k=1}^{n} k \mid p_{n-k} - p_{n-k-1} \mid = O(\mid P_n \mid), \text{ as } n \to \infty \tag{3.3}$$

If

$$\Psi(t) = o(1), \text{ as } t \to 0+, \tag{3.4}$$

then the sequence $\{n\, B_n(x)\}$ *is summable* $(N,p_n)\cdot C_1$ *to the value* ℓ/π.

4. Now we extend the above theorem to $||T||\cdot C_1$ - summability of the sequence $\{n\, B_n(x)\}$. We prove the following theorem:

Theorem: *Let* $||T|| \equiv (a_{n,k})$ *be an infinite triangular matrix with* $a_{n,k} \geq 0$ *with* $A_{n,0} = 1$, $\forall n \geq 0$ *and* $\{a_{n,k}\}_{k=0}^{n}$ *satisfy*

$$\sum_{1}^{n} k \mid a_{n,k} - a_{n,k+1} \mid = O(1), \text{ as } n \to \infty. \tag{4.1}$$

If

$$\Psi_1(t) = o(t) \text{ as } t \to 0+ \tag{4.2}$$

then the sequence $\{n\, B_n(x)\}$ *is summable* $||T||\cdot C_1$ *to the value* ℓ/π.

We note that condition (4.1) in the case of $(N,p_n)\cdot C_1$ summability reduce to condition (3.3), while condition (3.4) implies condition (4.2).

5. Proof of the Theorem:

If we denote the C_1 transformation of the sequence $\{n\, B_n(x)\}$ by σ_n, we have, after Mohanty and Nanda (1954), that

$$\sigma - \ell/\pi = \frac{1}{\pi}\int_0^{\pi} \Psi(t)[\frac{\sin nt}{nt^2} - \frac{\cos nt}{t}]dt + o(1).$$

Since the method of summability under consideration is regular, we have to show that under the conditions of our theorem.

$$\begin{aligned} I &= \int_0^{\pi} \frac{\Psi(t)}{\pi} \sum_{k=1}^{n} a_{n,k}(\frac{\sin kt}{kt^2} - \frac{\cos kt}{t})dt \\ &= o(1) \text{ as } n \to \infty. \end{aligned}$$

where

$$| I_{2,1} | \leq \sum_{k=N}^{n} | a_{n,k} - a_{n,k+1} | \int_{1/k}^{\delta} t^{-2} | \Psi(t) | dt$$

$$= \sum_{k=N}^{n} | a_{n,k} - a_{n,k+1} | \{[t^{-2}\Psi_1(t)]_{1/k}^{\delta} + \int_{1/k}^{\delta} t^{-3}\Psi_1(t)dt\}$$

$$= \sum_{N}^{n} | a_{n,k} - a_{n,k+1} | [o(t^{-1})_{1/k}^{\delta} + o(\int_{1/k}^{\delta} t^{-2}dt)]$$

$$= o[\sum_{N}^{n} k | a_{n,k} - a_{n,k+1} |] = o(1).$$

$$| I_{2,2} | \leq \sum_{k=N}^{n} a_{n,k+1} \int_{1/k+1}^{1/k} | \Psi(t) | t^{-2} dt$$

$$= \sum_{k=N}^{n} a_{n,k+1} [\{t^{-2}\Psi_1(t)\}_{1/k+1}^{1/k} + \int_{1/k+1}^{1/k} t^{-3}\Psi_1(t)dt]$$

$$= \sum_{N}^{n} a_{n,k+1} [o(t^{-1})_{1/k+1}^{1/k} + o(\int_{1/k+1}^{1/k} t^{-2}dt)]$$

$$= o(\sum_{N}^{n} \frac{| a_{n,k+1} |}{k(k+1)} = o(1).$$

Having fixed δ we are to show that $I_3 \to 0$ as $n \to \infty$. But this follows by Riemann-Lebesgue theorem and regularity of the method.

This proves the theorem.

References

[1] Khare,S.P. and Singh,Man, "A study of the sequence of Fourier Coefficients", *Bull. Cal. Math. Soc.* 80, (1988) 161.

[2] Lal,S.N., "On the Nörlund Summability of Fourier Series and the behaviour of Fourier Coefficient", *Indian Jl. Math.* 13, (1971) 177.

[3] Mohanty,R. and Nanda,M., "On the behaviour of Fourier Coefficients", *Proc. Amer. Math. Soc.* 5, (1954) 79.

[4] Sharma,R.M., "On $(N, p_n).C_1$ summability of the sequence $\{nB_n(x)\}$", *Rend. Circ. Palermo* 2, 19, (1970) 217.

[5] Varshney,O.P., "On a sequence of Fourier Coefficients", *Proc. Amer. Math. Soc.* 10, (1959) 790.

Department of Mathematics, University of Roorkee, Roorkee-247 667, India.

Assume that, $\exists\, \delta\, (0 < t \leq \delta)$:

$a_{n,m} = 0$ for every positive integer $m \leq [1/\delta]$.

Therefore

$$I = \int_0^\pi \frac{1}{\pi} \sum_{k=N}^{n} a_{n,k} \Psi(t)\, h_k(t) dt,$$

where $N = [1/\delta] + 1$ and

$$h_k(t) = \frac{\sin kt}{kt^2} - \frac{\cos kt}{t}$$

Let us write

$$\begin{aligned} \pi I &= \sum_{k=N}^{n} a_{n,k} \int_0^\pi \Psi(t)\, h_k(t) dt \\ &= \sum_{N}^{n} a_{n,k} [\int_0^{1/k} + \int_{1/k}^{\delta} + \int_\delta^\pi) \Psi(t) h_k(t) dt \\ &= I_1 + I_2 + I_3\,, \qquad \text{say.} \end{aligned}$$

Now

$$\begin{aligned} |\, I_1 \,| &= |\sum_{N}^{n} a_{n,k} \int_0^{1/k} \Psi(t)\, h_k(t) dt\, | \\ &\leq \sum_{N}^{n} a_{n,k} \int_0^{1/k} k^2 t \,|\, \Psi(t) \,|\, dt \\ &\leq \sum_{N}^{n} k\, a_{n,k} \int_0^{1/k} |\, \Psi(t) \,|\, dt \\ &= o(\sum_{N}^{n} a_{n,k}) = o(1), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{1}{\pi} \sum_{N}^{n} a_{n,k} \int_{1/k}^{\delta} \Psi(t)\, h_k(t) dt \\ &= \frac{1}{\pi} \sum_{N}^{n} a_{n,k} \int_{1/k}^{\delta} \Psi(t) [H_k(t) - H_{k-1}(t)] dt \end{aligned}$$

where

$$\begin{aligned} H_k(t) &= \frac{1}{t^2} \sum_{m=1}^{k} \frac{\sin mt}{m} - \frac{1}{t} \sum_{1}^{k} \cos mt \\ &= O(t^{-2}) \text{ for } \pi \geq t > 0 \text{ as easily seen.} \end{aligned}$$

Thus

$$\begin{aligned} I_2 &= \frac{1}{\pi} \sum_{k=N}^{n} (a_{n,k} - a_{n,k+1}) \int_{1/k}^{\delta} \Psi(t)\, H_k(t) dt - \frac{1}{\pi} \sum_{k=N}^{n} a_{n,k+1} \int_{1/(k+1)}^{1/k} \Psi(t)\, H_k(t) dt \\ &= I_{2,1} + I_{2,2}\,, \text{ say,} \end{aligned}$$