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INTERPOLATIONS OF DETERMINANTAL INEQUALITIES OF JENSEN'S TYPE

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As in [1] and [2] we shall use the following notation:

$\mathcal{M} =$ $\{M|M$ is a positive definite matrix of order $n\}$,

$|M| =$ the determinant of the matrix M,

$|M|_k = \Pi_{j=1}^k \lambda_j$, $k = 1, \cdots, n$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M with $\lambda_1 \leq \ldots \leq \lambda_n$, $|M|_n = |M|$,

$M(j) =$ the submatrix of M obtained by deleting the j^{th} row and column of M,

$M[k] =$ the principal submatrix of M formed by taking the first k rows and columns of M, $M[n] = M$, $M[n-1] = M(n)$, $M[0] =$ the identity matrix,

BBF= the class of Bellman-Bergstrom-Fan quasi-linear functionals σ_i, δ_j, and ν_k defined on $\mathcal{M}$ by

$$\sigma_i(M) = | M |_i^{1/i}, \; i = 1, \cdots, n,$$

$$\delta_j(M) = | M | / | M(j) |, \; j = 1, \cdots, n,$$

and

$$\nu_k(M) = (| M | / | M[k] |)^{1/(n-k)}, \; k = 1, \cdots, n,$$

respectively. It is evident that $\mathcal{M}$ is closed under addition and multiplication by a positive number, i.e. if $M_1, M_2 \in \mathcal{M}$, $a > 0$, then

$$M_1 + M_2, \; aM_1 \in \mathcal{M}.$$

Now, quasi-linearity of the BBF functionals follows from results given in [3, pp. 67, 70, 71] (see also [1,2]), i.e.

$$\phi(pM_1 + qM_2) \geq p\phi(M_1) + q\phi(M_2)$$

for $M_1, M_2 \in \mathcal{M}$, $p, q > 0$, $\phi \in BBF$.

More generaly, for $M_i \in \mathcal{M}$, $p_i > 0$ $(i = 1, \cdots, m)$, $P_k = \Sigma_{i=1}^k p_i$ $(k = 1, \cdots, m)$, $\phi \in BBF$, we have [2]:

$$\phi(\sum_{i=1}^m p_i M_i) \geq \sum_{i=1}^m p_i \phi(M_i) \geq P_m \prod_{i=1}^m \phi(M_i)^{p_i/P_m}, \tag{1}$$

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what is an interpolating inequality for

$$\phi\Big(\frac{1}{P_m}\sum_{i=1}^{m} p_i M_i\Big) \geq \prod_{i=1}^{m} \phi(M_i)^{p_i/P_m}. \tag{2}$$

Note that (2) is also a generalization of a result from [4].

For $p_i = 1/m$, (1) and (2) become

$$\phi\Big(\frac{1}{m}\sum_{i=1}^{m} M_i\Big) \geq \frac{1}{m}\sum_{i=1}^{m}\phi(M_i) \geq \prod_{i=1}^{m}\phi(M_i)^{1/m}, \tag{3}$$

and

$$\phi\Big(\frac{1}{m}\sum_{i=1}^{m} M_i\Big) \geq \prod_{i=1}^{m}\phi(M_i)^{1/m}. \tag{4}$$

Here, we shall give some interpolations of inequalities (3) and (4). In there results we shall use the following expresions:

$$f_{k,m} = \frac{1}{\binom{m}{k}} \sum_{1\leq i_1<\ldots<i_k\leq m} \phi\Big(\frac{1}{k}(M_{i_1}+\cdots+M_{i_k})\Big),$$

$$g_{k,m} = \prod_{1\leq i_1<\ldots<i_k\leq m} \Big(\frac{1}{k}(\phi(M_{i_1})+\cdots+\phi(M_{i_k}))\Big)^{1/\binom{m}{k}}$$

$$h_{k,m} = \frac{1}{\binom{m}{k}} \sum_{1\leq i_1<\ldots<i_k\leq m} (\phi(M_{i_1})\cdots\phi(M_{i_k}))^{1/k}$$

$$r_{k,m} = \prod_{1\leq i_1<\ldots<i_k\leq m} \phi\Big(\frac{1}{k}(M_{i_1}+\ldots+M_{i_k})\Big)^{1/\binom{m}{k}}.$$

First, we shall prove the following interpolations of the first inequality in (3):

$$\phi\Big(\frac{1}{m}\sum_{i=1}^{m} M_i\Big) = f_{m,m} \geq \ldots \geq f_{k+1,m} \geq f_{k,m} \geq \ldots \geq f_{1,m} = \frac{1}{m}\sum_{i=1}^{m}\phi(M_i). \tag{5}$$

Proof. We have

$$\begin{aligned} f_{k+1,m} &= \frac{1}{\binom{m}{k+1}} \sum_{1\leq i_1<\cdots<i_{k+1}\leq m} \phi\Big(\frac{M_{i_1}+\cdots+M_{i_{k+1}}}{k+1}\Big) \\ &\geq \frac{1}{\binom{m}{k+1}} \sum_{1\leq i_1<\cdots<i_{k+1}\leq m} \frac{1}{k+1}\sum_{j=1}^{k+1}\phi\Big(\frac{M_{1_1}+\cdots+M_{i_{k+1}}-M_{i_j}}{k}\Big) \\ &= \frac{1}{\binom{m}{k}} \sum_{1\leq i_1<\cdots<i_k\leq m} \phi\Big(\frac{M_{i_1}+\cdots+M_{i_k}}{k}\Big) = f_{k,m}. \end{aligned}$$

The second inequality in (3) has the following interpolations:

$$\frac{1}{m}\sum_{i=1}^{m}\phi(M_i) = g_{m,m} \geq \ldots \geq g_{k+1,m} \geq g_{k,m} \geq \ldots \geq g_{1,m} = \prod_{i=1}^{m}\phi(M_i)^{1/m}, \quad (6)$$

$$\frac{1}{m}\sum_{i=1}^{m}\phi(M_i) = h_{1,m} \geq \ldots \geq h_{k,m} \geq h_{k+1,m} \geq \ldots \geq h_{m,m} = \prod_{i=1}^{m}\phi(M_i)^{1/m}. \quad (7)$$

Proof. Now, we have

$$\begin{aligned} g_{k+1,m} &= \prod_{1\leq i_1<\ldots<i_{k+1}\leq m} \left(\frac{\phi(M_{i_1})+\ldots+\phi(M_{i_{k+1}})}{k+1}\right)^{1/\binom{m}{k+1}} \\ &\geq \prod_{1\leq i_1<\ldots<i_{k+1}\leq m} \left(\prod_{i=1}^{k+1}\left(\frac{\phi(M_{i_1})+\ldots+\phi(M_{i_{k+1}})-\phi(M_{i_j})}{k}\right)^{\frac{1}{k+1}}\right)^{\frac{1}{\binom{m}{k+1}}} \\ &= \prod_{1\leq i_1<\ldots<i_k\leq m} \left(\frac{\phi(M_{i_1})+\ldots+\phi(M_{i_k})}{k}\right)^{1/\binom{m}{k}} = g_{k,m}, \end{aligned}$$

and

$$\begin{aligned} h_{k+1,m} &= \frac{1}{\binom{m}{k+1}} \sum_{1\leq i_1<\ldots<i_{k+1}\leq m} (\phi(M_{i_1})\ldots\phi(M_{i_{k+1}}))^{1/(k+1)} \\ &\leq \frac{1}{\binom{m}{k+1}} \sum_{1\leq i_1<\ldots<i_{k+1}\leq m} \frac{1}{k+1}\sum_{j=1}^{k+1}\left(\frac{\phi(M_{i_1})\ldots\phi(M_{i_{k+1}})}{\phi(M_{i_j})}\right)^{1/k} \\ &= \frac{1}{\binom{m}{k}} \sum_{1\leq i_1<\ldots<i_k\leq m} (\phi(M_{i_1})\ldots\phi(M_{i_k}))^{1/k} = h_{k,m}. \end{aligned}$$

Finally, similar interpolation of (4) is:

$$\phi\left(\frac{1}{m}\sum_{i=1}^{m}M_i\right) = r_{m,m} \geq \ldots \geq r_{k+1,m} \geq r_{k,m} \geq \ldots \geq r_{1,m} = \prod_{i=1}^{m}\phi(M_i)^{1/m}. \quad (8)$$

Proof

$$\begin{aligned} r_{k+1,m} &= \prod_{1\leq i_1<\ldots<i_{k+1}\leq m} \phi\left(\frac{M_{i_1}+\ldots+M_{i_{k+1}}}{k+1}\right)^{1/\binom{m}{k+1}} \\ &\geq \prod_{1\leq i_1<\ldots<i_{k+1}\leq m} \left(\prod_{j=1}^{k+1}\phi\left(\frac{M_{i_1}+\ldots M_{i_{k+1}}-M_{i_j}}{k}\right)^{\frac{1}{k+1}}\right)^{\frac{1}{\binom{m}{k+1}}} \\ &= \prod_{1\leq i_1<\ldots<i_k\leq m} \phi\left(\frac{1}{k}(M_{i_1}+\ldots+M_{i_k})\right)^{1/\binom{m}{k}} = r_{k,m}. \end{aligned}$$

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GENERALIZATIONS OF OPIAL-TYPE INEQUALITIES IN TWO VARIABLES

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Abstract. In this paper some generalizations of the existing Opial-type inequalities in two variables are established. Such integral inequalities have a wide range of applications in the study of differential and integral equations.

1. Introduction

The Opial-type inequalities are among a handful of celebrated integral inequalities which plays an important role in the study of many differential and integral equations. The first Opial-type inequality is the following one obtained by Opial.

Theorem ([5]). *If $f \in \mathcal{L}^1[0,h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0,h)$, then*

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4}\int_0^h |f'(x)|^2 dx .$$

Opial's result has been generalized to many different situations, most of which are for the case of 1 variable (eg; [2], [4], [7], [8]). In [3], several Opial-type inequalities in 2 variables have been established. In this paper, we shall further generalize the results obtained in [3] to more general situations. For other Opial-type inequalities in 2-variables, which are less general, one is referred to, e.g., [6], and the references cited there.

2. Main Results

In what follows we shall assume that f, g, f_1, g_1, f_{12}, and g_{12} are continuous real-valued functions on $[a,b]\times[c,d]$, where, as usual, the subscripts refer to partial derivatives. Let w be a positive continuous weight function on $[a,b]\times[c,d]$. Let r be any positive function on $[a,b]\times[c,d]$ with $r^{-1} \in \mathcal{L}^1([a,b]\times[c,d])$.

Theorem 1. *If $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ for all $(s,t) \in [a,b]\times[c,d]$*

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and if $w(s,t)$ is nonincreasing in each variable, then

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1} \cdot \int_a^b \int_c^d r^{-1}\,dtds \cdot$$
$$\int_a^b \int_c^d rw(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Theorem 2. *If $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ and $0 < A \leq w(s,t) \leq B$ for all $(s,t) \in [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \cdot \int_a^b \int_c^d r^{-1}\,dtds$$
$$\times \int_a^b \int_c^d rw(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Theorem 3. *If $f(a,t) = g(a,t) = f(b,t) = g(b,t) = f_1(s,c) = g_1(s,c) = f_1(s,d) = g_1(s,d) = 0$ and $0 < A \leq w(s,t) \leq B$ for all $(s,t) \in [a,b] \times [c,d]$, then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}\left[\frac{(b-a)(d-c)}{4}\right]^{2p+q-1}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \cdot M$$
$$\times \int_a^b \int_c^d rw(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$, where

$$M := \max\left\{ \int_a^{(a+b)/2} \int_c^{(c+d)/2} r^{-1}\,dtds,\ \int_a^{(a+b)/2} \int_{(c+d)/2}^d r^{-1}\,dtds\ , \int_{(a+b)/2}^b \int_c^{(c+d)/2} r^{-1}\,dtds,\ \int_{(a+b)/2}^b \int_{(c+d)/2}^d r^{-1}\,dtds \right\}$$

Before proving these theorems, we first define, for any $(s,t) \in [a,b] \times [c,d]$,

$$F(s,t) := \int_a^s \int_c^t w(u,v)^{q/(2(p+q))}|f_{12}(u,v)|^q\,dvdu$$

and

$$G(s,t) := \int_a^s \int_c^t w(u,v)^{q/(2(p+q))} |g_{12}(u,v)|^q \, dv du \, .$$

Clearly we have $F \geq 0$, $G \geq 0$.

Lemma 1. *For any $(s,t) \in [a,b] \times [c,d]$ and any real numbers $p \geq 0$, $q > 0$.*

$$\begin{aligned} &\int_a^s \int_c^t w^{q/(2(p+q))} (FG)^{p/q} [F|g_{12}|^q + G|f_{12}|^q] dv du \\ &\leq \frac{q}{2(p+q)} [(s-a)(t-c)]^{2p/q} \int_a^s \int_c^t r^{-1} dv du \\ &\quad \times \int_a^s \int_c^t rw \left(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)} \right) dv du \, . \end{aligned}$$

Proof: Since

$$\begin{aligned} F_1(s,t) &= \int_c^t w(s,v)^{q/(2(p+q))} |f_{12}(s,v)|^q \, dv \, , \\ G_1(s,t) &= \int_c^t w(s,v)^{q/(2(p+q))} |g_{12}(s,v)|^q \, dv \, , \\ F_2(s,t) &= \int_a^s w(u,t)^{q/(2(p+q))} |f_{12}(u,t)|^q \, du \, , \\ G_2(s,t) &= \int_a^s w(u,t)^{q/(2(p+q))} |g_{12}(u,t)|^q \, du \, , \end{aligned}$$

both F and G are nondecreasing in each variable. Hence

$$\begin{aligned} &\int_a^s \int_c^t w(u,v)^{q/(2(p+q))} [F(u,v)G(u,v)]^{p/q} \cdot \\ &\qquad [F(u,v)|g_{12}(u,v)|^q + G(u,v)|f_{12}(u,v)|^q] dv du \\ &\leq \int_a^s [F(u,t)G(u,t)]^{p/q} [F(u,t)G_1(u,t) + F_1(u,t)G(u,t)] du \\ &= \frac{q}{p+q} \int_a^s [F(u,t)^{(p+q)/q} G(u,t)^{(p+q)/q}]_1 \, du \\ &= \frac{q}{p+q} F(s,t)^{(p+q)/q} G(s,t)^{(p+q)/q} \\ &\leq \frac{q}{2(p+q)} [F(s,t)^{2(p+q)/q} + G(s,t)^{2(p+q)/q}] \, . \end{aligned} \tag{2}$$

Now by Hölder's inequality,

$$\begin{aligned} F(s,t)^{2(p+q)/q} &= \left[\int_a^s \int_c^t r^{-q/(2(p+q))} (rw)^{q/(2(p+q))} |f_{12}|^q \, dv du \right]^{2(p+q)/q} \\ &\leq \left[\int_a^s \int_c^t r^{-q/(2p+q)} dv du \right]^{(2p+q)/q} \cdot \left[\int_a^s \int_c^t rw |f_{12}|^{2(p+q)} dv du \right] \\ &\leq [(s-a)(t-c)]^{2p/q} \cdot \int_a^s \int_c^t r^{-1} dv du \cdot \int_a^s \int_c^t rw |f_{12}|^{2(p+q)} dv du \end{aligned}$$

and similarly,

$$G(s,t)^{2(p+q)/q} \le \left[(s-a)(t-c)\right]^{2p/q} \cdot \int_a^s \int_c^t r^{-1}\,dvdu \cdot \int_a^s \int_c^t rw|g_{12}|^{2(p+q)}\,dvdu\ .$$

The lemma now follows by plugging these back into (2). Q.E.D.

Proof of Theorem 1: Since w is nonincreasing in each variable, by Hölder's inequality we have

$$\begin{aligned}
|f(s,t)| &\le \int_a^s |f_1(u,t)|du \\
&\le \int_a^s \int_c^t |f_{12}(u,v)|dvdu \\
&\le w(s,t)^{-1/(2(p+q))} \int_a^s \int_c^t w(u,v)^{1/(2(p+q))}|f_{12}(u,v)|dvdu \\
&\le w(s,t)^{-1/(2(p+q))}[(s-a)(t-c)]^{(q-1)/q} F(s,t)^{1/q}
\end{aligned}$$

and similarly,

$$|g(s,t)| \le w(s,t)^{-1/(2(p+q))}[(s-a)(t-c)]^{(q-1)/q} G(s,t)^{1/q}$$

for any $(s,t) \in [a,b]\times[c,d]$. Hence by lemma 1,

$$\begin{aligned}
&\int_a^b \int_c^d w|fg|^p\left(|fg_{12}|^q + |f_{12}g|^q\right) dtds \\
&\le \int_a^b \int_c^d [(s-a)(t-c)]^{(2p+q)(q-1)/q} w^{q/(2(p+q))}(FG)^{p/q}\left(F|g_{12}|^q + G|f_{12}|^q\right) dtds \\
&\le [(b-a)(d-c)]^{(2p+q)(q-1)/q} \int_a^b \int_c^d w^{q/(2(p+q))}(FG)^{p/q}\left(F|g_{12}|^q + G|f_{12}|^q\right) dtds \\
&\le \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1} \cdot \int_a^b \int_c^d r^{-1}\,dvdu \\
&\quad \int_a^b \int_c^d rw\left(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)}\right) dtds\ .
\end{aligned}$$

Q.E.D

Proof of Theorem 2: Since $0 < A \le w \le B$ on $[a,b]\times[c,d]$, by Hölder's inequality we have,

$$\begin{aligned}
|f(s,t)| &\le \int_a^s \int_c^t |f_{12}(u,v)|dvdu \\
&\le A^{-1/(2(p+q))} \int_a^s \int_c^t w(u,v)^{1/(2(p+q))}|f_{12}(u,v)|dvdu \\
&\le A^{-1/(2(p+q))}[(s-a)(t-c)]^{(q-1)/q} F(s,t)^{1/q}
\end{aligned}$$

and similarly,

$$|g(s,t)| \le A^{-1/(2(p+q))}[(s-a)(t-c)]^{(q-1)/q}G(s,t)^{1/q}$$

for any $(s,t) \in [a,b] \times [c,d]$. Hence by lemma 1,

$$\begin{aligned}
&\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds \\
&\le \int_a^b \int_c^d A^{-(2p+q)/(2(p+q))}[(s-a)(t-c)]^{(2p+q)(q-1)/q} \\
&\quad w(FG)^{p/q}(F|g_{12}|^q + G|f_{12}|^q)\,dtds \\
&\le [(b-a)(d-c)]^{(2p+q)(q-1)/q}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \\
&\quad \int_a^b \int_c^d w^{q/(2(p+q))}(FG)^{p/q}(F|g_{12}|^q + G|f_{12}|^q)\,dtds \\
&\le \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \int_a^b \int_c^d r^{-1}\,dtds \\
&\quad \times \int_a^b \int_c^d r\,w(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds\,. \qquad Q.E.D.
\end{aligned}$$

Proof of Theorem 3. We first divide the rectangle $[a,b] \times [c,d]$ into four rectangles $[a,(a+b)/2] \times [c,(c+d)/2]$, $[a,(a+b)/2] \times [(c+d)/2,d]$, $[(a+b)/2,b] \times [c,(c+d)/2]$, and $[(a+b)/2,b] \times [(c+d)/2,d]$. By theorem 2, inequality (1) holds when restricted to the rectangle $[a,(a+b)/2] \times [c,(c+d)/2]$. Now by analogous statements of theorem 2 (with appropriate choices of the functions F and G), it is readily seen that inequality (1) also holds when restricted to the other three rectangles. Hence the theorem. Q.E.D.

By putting $r = w^{-1}$ in theorems 1, 2, and 3, respectively, we have

Corollary 1. *If $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ for all $(s,t) \in [a,b] \times [c,d]$ and if $w(s,t)$ is nonincreasing in each variable, then*

$$\begin{aligned}
&\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds \\
&\le \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1}\int_a^b \int_c^d w\,dtds \int_a^b \int_c^d (|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds
\end{aligned}$$

for any real numbers $p \ge 0$, $q \ge 1$.

Corollary 2. *If $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ and $0 < A \le w(s,t) \le B$*

for all $(s,t) \in [a,b] \times [c,d]$, *then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q-1}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \int_a^b \int_c^d w\,dtds$$
$$\times \int_a^b \int_c^d (|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Corollary 3. *If* $f(a,t) = g(a,t) = f(b,t) = g(b,t) = f_1(s,c) = g_1(s,c) = f_1(s,d) = g_1(s,d) = 0$ *and* $0 < A \leq w(s,t) \leq B$ *for all* $(s,t) \in [a,b] \times [c,d]$, *then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}\left[\frac{(b-a)(d-c)}{4}\right]^{2p+q-1}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))} \cdot M$$
$$\times \int_a^b \int_c^d (|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$, *where*

$$M := \max\left\{ \int_a^{(a+b)/2} \int_c^{(c+d)/2} w\,dtds,\ \int_a^{(a+b)/2} \int_{(c+d)/2}^d w\,dtds,\right.$$
$$\left. \int_{(a+b)/2}^b \int_c^{(c+d)/2} w\,dtds,\ \int_{(a+b)/2}^b \int_{(c+d)/2}^d w\,dtds \right\}.$$

By putting $r \equiv 1$ in theorems 1, 2, and 3, respectively, we have

Corollary 4. *If* $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ *for all* $(s,t) \in [a,b] \times [c,d]$ *and if* $w(s,t)$ *is nonincreasing in each variable, then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q} \int_a^b \int_c^d w\big(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)}\big)\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Corollary 5. *If* $f(a,t) = g(a,t) = f_1(s,c) = g_1(s,c) = 0$ *and* $0 < A \leq w(s,t) \leq B$

for all $(s,t) \in [a,b] \times [c,d]$, *then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}[(b-a)(d-c)]^{2p+q}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))}$$
$$\times \int_a^b \int_c^d w(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Corollary 6. *If* $f(a,t) = g(a,t) = f(b,t) = g(b,t) = f_1(s,c) = g_1(s,c) = f_1(s,d) = g_1(s,d) = 0$ *and* $0 < A \leq w(s,t) \leq B$ *for all* $(s,t) \in [a,b] \times [c,d]$, *then*

$$\int_a^b \int_c^d w|fg|^p(|fg_{12}|^q + |f_{12}g|^q)\,dtds$$
$$\leq \frac{q}{2(p+q)}\left[\frac{(b-a)(d-c)}{4}\right]^{2p+q}\left(\frac{B}{A}\right)^{(2p+q)/(2(p+q))}$$
$$\times \int_a^b \int_c^d w(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)})\,dtds$$

for any real numbers $p \geq 0$, $q \geq 1$.

Notice that corollaries 4, 5, and 6 are exactly theorems 1, 2, and 3, respectively, in [3]. In general, one would wish that the annoying term $(B/A)^{(2p+q)/(2(p+q))}$ in theorems 2 and 3 could be removed. In fact, in [1] it was claimed that this could be done for theorem 3 by requiring w to be a positive bounded function which is nonincreasing in each variable. However, the proof there contained several crucial mistakes. As a matter of fact, the following examples show that in general one could not remove the term $(B/A)^{(2p+q)/(2(p+q))}$ in either theorem 2 or theorem 3.

Example 1. Let $f(s,t) = g(s,t) = st$ on $[0,1] \times [0,1]$, $p = 0$, and $r = q = 1$. Then

$$\int_a^b \int_c^d w|fg|^p\left(|fg_{12}|^q + |f_{12}g|^q\right)dtds = 2\int_0^1 \int_0^1 w \cdot st\,dt\,ds\ , \tag{3}$$

while

$$\frac{q}{2(p+q)}\left[(b-a)(d-c)\right]^{2p+q}\int_a^b \int_c^d w\left(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)}\right)dtds = \int_0^1 \int_0^1 w\,dtds. \tag{4}$$

Observe that any positive continuous weight function $w(s,t)$ which is large in, say $[0.8,1] \times [0.8,1]$ and small elsewhere will make (3) > (4). Thus in general the term $(B/A)^{(2p+q)/(2(p+q))}$ in theorem 2 is not removable.

Notice that example 1 also shows that the nonincreasing property of the weight function w in theorem 1 is essential.

Example 2. Let $f(s,t) = g(s,t) = st(1-s)(1-t)$ on $[0,1] \times [0,1]$, $p = 0$, and $q = 1$. Then

$$\begin{aligned} &\int_a^b \int_c^d w|fg|^p \left(|fg_{12}|^q + |f_{12}g|^q\right) dtds \\ &= 2 \int_0^1 \int_0^1 w \cdot st(1-s)(1-t)|1-2s||1-2t|\, dtds\ , \end{aligned} \tag{5}$$

while

$$\begin{aligned} &\frac{q}{2(p+q)} \left[\frac{(b-a)(d-c)}{4}\right]^{2p+q} \int_a^b \int_c^d w\left(|f_{12}|^{2(p+q)} + |g_{12}|^{2(p+q)}\right) dtds \\ &= \frac{1}{4} \int_0^1 \int_0^1 w|1-2s|^2|1-2t|^2\, dtds\ . \end{aligned} \tag{6}$$

Now it is easy to check that in $D = [0.4, 0.6] \times [0.4, 0.6]$, we have $s(1-s) > |1-2s|$, $t(1-t) > |1-2t|$. Thus for any positive continuous weight function $w(s,t)$ which is sufficiently large in D and small elsewhere, we have (5) > (6). Hence in general the term $(B/A)^{(2p+q)/(2(p+q))}$ in theorem 3 is essential.

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