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UNIFIED TREATMENT OF LOCAL THEORY OF NFDES WITH INFINITE DELAY

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Abstract. The present paper gives a unified treatment of the local theory of NFDEs with infinite delay on a class of comparatively comprehensive phase spaces which contain admissible phase spaces and BC space as special cases. Conditions and assumptions are imposed on two functionals defining the equation, and therefore independent of the structure and properties of phase spaces. This allows us to determine phase spaces and sufficient conditions guaranteeing existence and uniqueness of solutions according to the property of the equation at hand as opposed to preassign a phase space to dictate the conditions. An example will be given to show how to choose the phase space according to the fading memory characteristic of Volterra integrodifferential equations so that the Cauchy initial value problem is well posed.

1. Introduction

Local theory has been investigated in various literatures under various assumptions and formulations for functional differential equations(FDE) with infinite delay

$$\dot{x}(t) = f(t, x_t) \tag{1.1}$$

where $x_t(s) = x(t+s), -\infty < s \leq 0$, f is a map defined on a certain product space $R \times S$ with S denoting a function space whose elements are defined on $(-\infty, 0]$ and the values in R^n. Existence, uniqueness, continuous dependence and continuation theorems have been obtained for a variaty of phase spaces provided adequate conditions are imposed on the right-hand side functional f.

In order to deal with equations with infinite delay on a large variaty of phase spaces, Hale and Kato [15] and Schmacher [19] independently developed a general theory which

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has the feature of axiomatic approach-to list certain axioms for the phase space and the right-hand side functional of (1.1), such that any particular space and $f(t, \phi)$ verify their axioms, automatically generate existence and uniqueness. While the progress in the qualitative analysis of equations with infinite delay in the framework of axiomatic approach has been overwhelming during the last decade, it should be pointed out that not all the usual phase spaces are included in those so called admissible phase spaces described in [15] and [19]. For example, the basic axioms exclude the space(BC) of bounded continuous functions which has been proven to be convenient and useful for the investigation of many Volterra integrodifferential equations and integral equations(see, eg, Burton [3],[4] and Miller [17]). The discontinuity of $t \to x_t \in BC$ with respect to the super-norm enforces investigators to deal with many Volterra integrodifferential equations and integral equations in a different way than axiomatic approach. It is interesting to point out that while there is a strong parallel between the theory in the framework of axiomatic approach and the theory for Volterra integrodifferential equations, methods and results in these two theories are independent since they heavily rely on the structure and properties of the considered phase spaces.

The main purpose of this paper is to present a unified treatment for the local theory of FDEs with infinite delay on a calss of comparatively comprehensive phase spaces which contain admissible phase spaces and BC space as special cases. Conditions and assumptions are imposed on functionals defining the equatiuon, and therefore independent of the structure and properties of phase spaces. This allows us to determine phase spaces and sufficient conditions guaranteeing existence and uniqueness of solutions according to the property of the equation at hand as opposed to preassign a phase space to dictate the conditions.

The class of equations we will investigate in this paper is of the form

$$\frac{d}{dt}D(t, x_t) \ = \ f(t, x_t) \tag{1.2}$$

where D and f are functionals defined on R^n. This class of equation is called a neutral functional differential equation with infinite delay(NFDE) which contains as a special case the retarded FDEs described in most literatures(eg. Hale-Kato[15] and Schmacher [19]). Equation (1.2) contains also some Volterra integrodifferential equations depending on the history of derivative as following

$$\frac{d}{dt}[x(t) - \int_0^t G(t,s)x(s)ds] \ = \ Q(t)x(t) + \int_0^t C(t,s)x(s)ds$$

which arising from the investigation of stability of the following Volterra integrodifferential equation

$$\dot{x}(t) \ = \ A(t)x(t) + \int_0^t E(t,s)x(s)ds$$

in the case where $A(t)$ may vanish at any $t \geq 0$ (see, eg. Burton-Mahfoud [5] and Arino-Burton-Haddock [1]), and comes out in a very natural manner if one searches for the limit behaviors of solutions to Volterra equations with finite limit (see eg. Wu [23] and [24]).

In [20]-[22], the local theory of neutral equations with infinite delay was established on admissible phase spaces, BC and infinite dimensional phase spaces. One of the motivations for writing this paper comes from droping off the Frechet differentiability condition of functionals defining neutral equations required in [20]-[22]. This condition is not satisfied by the following Volterra integrodifferential equation

$$\frac{d}{dt}[x(t)-\sum_{k=1}^{\infty}B_k(t,x(t-r_k))-\int_{-\infty}^{t}G(t,s,x(s))ds]$$
$$=\sum_{k=1}^{\infty}A_k(t,x(t-r_k))+\int_{-\infty}^{t}H(t,s,x(s))ds \tag{1.1}$$

if the involved functions B_k and G are continuous but not continuously differentiable.

The paper will be organized as follows. In Section 2, we will give general discription of phase spaces of NFDEs with infinite delay. Section 3 is contributed to the establishment of the general local theory on comparatively compressive phase spaces. In Section 4, we will show how to choose the phase space according to the fading memory characteristic of Volterra integrodifferential equations so that general assumptions in Section 3 are satisfied, and therefore, the Cauchy initial value problem is well posed.

2. Phase Spaces and NFDEs with Infinite Delay

Let $|.|$ denote an R^n-norm, B be a real vector space either

(i) of continuous functions that map $(-\infty,0]$ to R^n with $\phi=\psi$ if $\phi(s)=\psi(s)$ on $(-\infty,0]$, or

(ii) Of measurable functions that map $(-\infty,0]$ to R^n with $\phi=\psi$(or ϕ is equivalent to ψ) in B if $\phi(s)=\psi(s)$ almost everywhere on $(-\infty,0]$, and $\phi(0)=\psi(0)$.

Let B be endowed with a norm $||.||_B$ such that B is completed with respect to $||.||_B$. Thus B equipped with norm $||.||_B$ is a Banach space. We denote this space by $(B,||.||_B)$ or simply by B, whenever no confusion can result.

If $x:(-\infty,A)\to R^n, 0\le A\le\infty$, then for any $t\in[0,A)$ define x_t by $x_t(s)=x(t+s)$ for $s\le 0$. Throghout this paper, suppose that phase space B satisfies the following condition:

Let $0\le a<A$. If $x:(-\infty,A)\to R^n$ is given such that $x_a\in B$ and $x\in[a,A)\to R^n$ is continuous, then $x_t\in B$ for all $t\in[a.A)$.

This is a very weak condition that the common admissible phase spaces (see, cf. [4]) and BC satisfy.

We will consider neutral differential equations with infinite delay of the following form

$$\frac{d}{dt}D(t,x_t) = f(t,x_t) \tag{2.1}$$

where $t \in [0,\infty), \Omega$ is an open set in B and $D, f : [0,\infty) \times \Omega \to R^n$. A typical example of NFDEs with infinite delay is the following Volterra integrodifferential equation

$$\frac{d}{dt}[x(t) - \sum_{k=1}^{\infty} B_k(t, x(t-r_k)) - \int_{-\infty}^{t} G(t,s,x(s))ds]$$
$$= \sum_{k=0}^{\infty} A_k(t, x(t-r_k)) + \int_{-\infty}^{t} G(t,s,x(s))ds$$

An initial condition of the form

$$x_{t_0} = \varphi, (t_0, \phi) \in [0, \phi) \times \Omega \tag{2.2}$$

will be considered so that (2.1)-(2.2) will comparise a typical Cauchy initial value problem(IVP).

Definition 2.1: A function $x : (-\infty, t_0 + \delta) \to R^n (t_0 \in [0,\infty), \delta > 0)$ is said to be a *solution* of the (IVP) (2.1)-(2.2) through (t_0, ϕ) on $[t_0, t_0 + \delta)$, if

(i) $x_{t_0} = \phi$,
(ii) x is continuous on $[t_0, t_0 + \delta)$,
(iii) $D(t, x_t)$ is absolutely continuous on $[t_0, t_0 + \delta)$,
(iv) (2.1) holds almost everywhere on $[t_0, t_0 + \delta)$.

According to this definition, (IVP) (2.1)-(2.2) is equivalent to the following equation

$$D(t, x_t) = D(t_0, x_{t_0}) + \int_{t_0}^{t} f(s, x_s)ds \tag{2.3}$$

for $t \in [t_0, t_0 + \delta)$ under some integration assumptions on $f(t, x_t)$ and some continuity assumptions on $D(t, x_t)$ which will be illustrated in next section.

3. Local Theory

Let $\Omega \subseteq B$ be an open set such that for any $(t_0, \phi) \in [0,\infty) \times \Omega$ there exist constants $\delta, \gamma > 0$ so that $x_t \in \Omega$ provided that $x \in A(t_0, \phi, \delta, \gamma)$ and $t \in [t_0, t_0 + \delta]$, where $A(t_0, \phi, \delta, \gamma)$ is defined as

$$A(t_0, \phi, \delta, \gamma) = \{x : (-\infty, t_0 + \delta] \to R^n, x_{t_0} = \phi, \sup_{t_0 \le t \le t_0 + \delta} | x(t) - \phi(0) | \le \gamma\}$$

Definition 3.1: $D : [0,\infty) \times \Omega \to R^n$ is called to be *generalized atomic* on Ω, if

$$D(t,\phi) - D(t,\psi) = K(t,\phi,\psi)[\phi(0) - \psi(0)] + L(t,\phi,\psi) \tag{3.1}$$

where $(t,\phi,\psi) \in [0,\infty) \times \Omega \times \Omega, K : [0,\infty) \times \Omega \times \Omega \to R^{n \times n}$ and $L : [0,\infty) \times \Omega \times \Omega \to R^n$ satisfy

(a) $\det K(t,\phi,\varphi) \neq 0$ for all $(t,\phi) \in [0,\infty) \times \Omega$,

(b) for any $(t_0,\phi) \in [0,\infty) \times \Omega$, there exist constants $\delta, \gamma > 0$ and $k_1, k_2 > 0$ with $2k_2 + k_1 < 1$ such that for all $x, y \in A(t_0,\phi,\delta,\gamma)$, $D(t,x_t)$, $K(t,x_t,y_t)$ and $L(t,x_t,y_t)$ are continuous in $t \in [t_0, t_0+\delta]$, and

$$| K^{-1}(t_0,\phi,\phi)L(t,x_t,y_t) | \leq k_1 \sup_{t_0 \leq s \leq t} | x(s) - y(s) | \tag{3.2}$$

$$| K^{-1}(t_0,\phi,\phi)K(t,x_t,y_t) - I | \leq k_2 \tag{3.3}$$

where I is an $n \times n$ unit matrix.

Theorem 3.1 (Existence): *If D is generalized atomic on Ω, and if for any $(t_0,\phi) \in [0,\infty)\times\Omega$, there exist constants $\delta, \gamma > 0$ and an integrable function $m : [t_0, t_0+\delta] \to [0,\infty)$ such that*

(c) *for any $x \in A(t_0,\phi,\delta,\gamma), f(t,x_t)$ is measurable and $|f(t,x_t)| \leq m(t)$ for $t \in [t_0, t_0+\delta]$,*

(d) *for any $x, y \in A(t_0,\phi,\delta,\gamma)$ with $\sup_{t_0 \leq s \leq t_0} + \delta|x(s) - y(s)| \to 0$ we have*

$$\int_{t_0}^{t_0+\delta} | f(s,x_s) - f(s,y_s) | \, ds \to 0$$

Then $(IVP)(2.1) - (2.2)$ has a solution.

Proof. For any $(t_0,\phi) \in [0,\infty) \times \Omega$ and constants $\delta, \gamma > 0$ define $E(\delta,\gamma)$ as follows

$$E(\delta,\gamma) = \{z : (-\infty, \delta] \to R^n \text{ is continuous}; z(s) = 0 \text{ for } s \in (-\infty, 0] \text{ and } ||z|| \leq \gamma\}$$

where $||z|| = \sup_{0 \leq s \leq \delta} |z(t)|$. Then $E(\delta,\gamma)$ with the norm $||.||$ is a Banach space. Choose δ and $\gamma > 0$ sufficiently small so that (a)-(d) hold, and thus there exist $\delta, \gamma > 0$ and positive constant $\alpha > 1$ such that

$$\alpha(2k_2 + k_1) < 1 \tag{3.4}$$

$$| K^{-1}(t_0,\phi,\phi)K(t,x_t,y_t) - I | \leq k_2 \tag{3.5}$$

$$| I - K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi) | \leq \min\{\alpha k_2, \alpha - 1\} \tag{3.6}$$

$$\int_{t_0}^{t_0+\delta} m(s)ds \leq \frac{1-\alpha(2k_2+k_1)}{2\alpha \, | K^{-1}(t_0,\phi,\phi) |}\gamma \tag{3.7}$$

where $\hat{\phi} \in A(t_0,\phi,\delta,\gamma)$ is defined as $\hat{\phi}_0 = \phi$ and $\hat{\phi}(t) = \phi(0)$ for all $t \in [0,\delta]$. Now, on $E(\delta,\gamma)$ define two operators S and U as follows

$$(Sz)(t) = \begin{cases} 0 \text{ for } t \leq 0 \\ K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[-D(t_0+t,\hat{\phi}_t+z_t) + D(t_0,\phi) + K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)] \\ \text{for } t \in [0,\delta] \end{cases}$$

and

$$(Uz)(t) = \begin{cases} 0 \text{ for } t \le 0 \\ K^{-1}(t_0+t,\hat{\phi},\hat{\phi}_t)\int_0^t f(t_0+s,\hat{\phi}_s+z_s)ds \text{ for } t\in[0,\delta] \end{cases}$$

for $z \in E(\delta,\gamma)$. Obviously, $(Sz)(t)$ and $(Uz)(t)$ are continuous in $t \in [0,\delta]$, and for $t\in[0,\delta]$, we have

$$\begin{aligned}
&\mid (Uz)(t) \mid \\
&= \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)K^{-1}(t_0,\phi,\phi)\int_0^t f(t_0+s,\hat{\phi}_s+z_s)ds \mid \\
&\le \alpha \mid K^{-1}(t_0,\phi,\phi)\int_0^t f(t_0+s,\hat{\phi}_s+z_s)ds \mid \\
&\le \alpha \mid K^{-1}(t_0,\phi,\phi)\mid \int_{t_0}^{t_0+\delta} m(s)ds \\
&\le \frac{1-\alpha(2k_2+k_1)}{2}\gamma
\end{aligned}$$

and

$$\begin{aligned}
&\mid (Sz)(t) \mid \\
&= \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[-D(t_0+t,\hat{\phi}_t+z_t)+D(t_0+t,\hat{\phi}_t) \\
&\qquad -D(t_0+t,\hat{\phi}_t)+D(t_0,\phi)+K(t_0+t,\hat{\phi},\hat{\phi}_t)z(t) \mid \\
&= \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[-K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)z(t)-L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t) \\
&\qquad -D(t_0+t,\hat{\phi}_t)+D(t_0,\phi)+K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)z(t) \mid \\
&= \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)-K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)]z(t) \\
&\qquad +K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[-L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)-D(t_0+t,\hat{\phi}_t)+D(t_0,\phi)] \mid \\
&= \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)[K^{-1}(t_0,\phi,\phi)K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)-I]z(t) \\
&\qquad -K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)[K^{-1}(t_0,\phi,\phi)K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)-I]z(t) \\
&\qquad +K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[-L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)-D(t_0+t,\hat{\phi}_t)+D(t_0,\phi)] \mid \\
&\le \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)\mid [\mid K^{-1}(t_0,\phi,\phi)K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)-I\mid \\
&\qquad +\mid K^{-1}(t_0,\phi,\phi)K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)-I\mid]\mid z(t)\mid \\
&\qquad +\mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t)\mid \\
&\qquad +\mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)\mid\mid D(t_0+t,\hat{\phi}_t)-D(t_0,\phi)\mid \\
&\le \mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)\mid [2k_2+k_1]\mid z(t)\mid \\
&\qquad +\mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)\mid\mid D(t_0+t,\hat{\phi}_t)-D(t_0,\phi)\mid
\end{aligned}$$

Now $D(t_0+t,\hat{\phi}_t)$ is continuous at $t=0$, so we can find $\delta>0$ sufficiently small so that

$$\mid K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)\mid\mid D(t_0+t,\hat{\phi})-D(t_0,\phi)\mid \le \frac{1-\alpha(2k_2+k_1)}{2}\gamma$$

and thus

$$|(Sz)(t)| \leq \alpha(2k_2+k_1)\gamma + \frac{1-\alpha(2k_2+k_1)}{1}\gamma = \frac{1+\alpha(2k_2+k_1)}{2}\gamma$$

Therefore, $|(Sz)(t)+(Uz)(t)| \leq \gamma$ for $t \in [0,\delta]$. This means that $S+U$ is a mapping from $E(\delta,\gamma)$ into itself.

On the other hand, for any $z, w \in E(\delta,\gamma)$, we have

$$\begin{aligned}
&|(Sz)(t)-(Sw)(t)| \\
\leq\ & |K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)[K(t_0+t,\hat{\phi}_t,\hat{\phi}_t)-K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t+w_t)]|\,|z(t)-w(t)| \\
&+|K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t+w_t)| \\
\leq\ & |[I-K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)]-K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi) \\
&\times[K^{-1}(t_0,\phi,\phi)K(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t+w_t)-I]|\,|z(t)-w(t)| \\
&+|K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)K(t_0,\phi,\phi)L(t_0+t,\hat{\phi}_t+z_t,\hat{\phi}_t+w_t)| \\
\leq\ & [\alpha+\alpha]k_2\,|z(t)-w(t)| + \alpha k_1 \sup_{0\leq s\leq t}|z(s)-w(s)| \\
\leq\ & \alpha(2k_2+k_1)\sup_{0\leq s\leq t}|z(s)-w(s)|
\end{aligned}$$

$\alpha(2k_2+k_1)<1$, therefore S is a contraction mapping on $E(\delta,\gamma)$.

For any $t,\tau \in [0,\delta]$, we have

$$\begin{aligned}
&|(Uz)(t)-(Uz)(\tau)| \\
=\ & |K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)\int_0^t f(t_0+s,\hat{\phi}_s+z_s)ds \\
&-K^{-1}(t_0+\tau,\hat{\phi}_\tau,\hat{\phi}_\tau)\int_0^\tau f(t_0+s,\hat{\phi}_s+z_s)ds| \\
\leq\ & |K^{-1}(t_0+t,\hat{\phi}_t,\hat{\phi}_t)-K^{-1}(t_0+\tau,\hat{\phi}_\tau,\hat{\phi}_\tau)|\int_{t_0}^{t_0+\delta} m(s)ds \\
&+|K^{-1}(t_0+\tau,\hat{\phi}_\tau,\hat{\phi}_\tau|\int_{t_0}^{t_0+(t-\tau)} m(s)ds
\end{aligned}$$

This means that $\{Uz; z\in E(\delta,\gamma)\}$ is equicontinuous, and thus U is a completely continuous operator by Ascoli-Arzela Theorem. Therefore $S+U$ is a α-contraction mapping on $E(\delta,\gamma)$, Darbo's fixed point theorem (see eg., [13, Theorem 6.3, pp98]) shows that (IVP) (2.1)-(2.2) has a solution $x(t)=\phi(0)+z(t-t_0)$ for all. $t\in[t_0,t_0+\delta]$. This completers the proof.

Theorem 3.2(Uniqueness): *Suppose that all conditions of Theorem 3.1 hold. Moreover, suppose that for any $(t_0,\phi)\in[0,\infty)\times\Omega$, there exist constants $\delta,\gamma>0$ and a* *nonnegative function* $g : [0, \delta] \to [0, \infty)$ *continuous at* $t = 0$ *and* $g(0) = 0$ *such that for any* $x, y \in A(t_0, \phi, \delta, \gamma)$ *we have*

$$| \int_{t_0}^{t} [f(s, xs) - f(s, y_s)]ds | \leq g(t - t_0) \sup_{t_0 \leq s \leq t} | x(s) - y(s) |$$

Then the solution of (IVP) (2.1) – (2.2) *is unique.*

Proof. According to the argument of Theorem 3.1, it suffices to prove that $S + U$ has a unique fixed point on $E(\delta, \gamma)$. Now, choose $\delta > 0$ sufficiently small so that

$$\sup_{0 \leq s \leq \delta} | K^{-1}(t_0 + s, \hat{\phi}_s, \hat{\phi}_s) | \sup_{0 \leq s \leq \delta} | g(s) | + \alpha(2k_2 + k_1) < 1$$

If y and z are both fixed points of $S + U$ on $E(\delta, \gamma)$, then using the same argument as that of Theorem 3.1, we get

$$\begin{aligned} &| (Sz)(t) - (Sy)(t) | \\ &\leq \alpha(2k_2 + k_1) \sup_{0 \leq s \leq t} | z(t) - y(t) | \end{aligned}$$

and

$$\begin{aligned} &| (Uz)(t) - (Uy)(t) | \\ = \ &| K^{-1}(t_0 + t, \hat{\phi}_t, \hat{\phi}_t)[\int_0^t f(t_0 + s, \hat{\phi}_s + z_s)ds - \int_0^t f(t_0 + s, \hat{\phi}_s + y_s)ds] | \\ \leq \ &| K^{-1}(t_0 + t, \hat{\phi}_t, \hat{\phi}_s) || g(t) | \sup_{0 \leq s \leq t} | z(s) - y(s) | \\ \leq \ &\sup_{0 \leq s \leq \delta} | K^{-1}(t_0 + s, \hat{\phi}_s, \hat{\phi}_s) || g(s) | \sup_{0 \leq s \leq t} | z(s) - y(s) | \end{aligned}$$

Therefore

$$\begin{aligned} &| z(t) - y(t) | \\ \leq \ &| (S + U)z(t) - (S + U)y(t) | \\ \leq \ &[\alpha(2k_2 + k_1) + \sup_{0 \leq s \leq \delta} | K^{-1}(t_0 + s, \hat{\phi}_s, \hat{\phi}_s) || g(s) |] \sup_{0 \leq s \leq t} | z(s) - y(s) | \end{aligned}$$

Therefore $z(t) = y(t)$ on $[0, \delta]$ since

$$\alpha(2k_2 + k_1) + \sup_{0 \leq s \leq \delta} | K^{-1}(t_0 + s, \hat{\phi}_s, \hat{\phi}_s) || g(s) | < 1$$

This completes the proof.

Throughout the remainder of this paper, for any $(t_0, \phi) \in [0, \infty) \times \Omega, \omega \subseteq \Omega$ and positive constants $\delta, \gamma > 0$, define $B_\omega(t_0, \phi, \delta, \gamma)$ as the set of all maps $x : (-\infty, t_0 + \delta] \to$

R^n such that $x_{t_0} = \phi, x : [t_0, t_0 + \delta) \to R^n$ is continupus with $|x(t)| \leq \gamma$ and $x_t \in \omega$ for all $t \in [t_0, t_0 + \delta)$. In the following theorem, F is a set of all subsets of Ω such that for any $(t_0, \phi) \in [0, \infty) \times \Omega$, constants $\delta, \gamma > 0$ and a set $\omega \in F$, if $x \in B_\omega(t_0, \phi, \delta, \gamma)$ and if $x(t_0 + \delta) := \lim_{t \to (t_0+\delta)} -x(t)$ exists, than $x_{t_0+\delta} \in \Omega$.

Theorem 3.3(Continuation): *Let all conditions of Theorem 3.1 hold. Besides, suppose that for any $B_\omega(t_0, \phi, \delta, \gamma)$ and any $x \in B_\omega(t_0, \phi, \delta, \gamma)$, we have*

(i) *there exists a locally integrable function $m : [t_0, t_0+\delta] \to [0, \infty)$ such that $f(t, x_t)$ is measurable and $|f(t, x_t)| \leq m(t)$,*

(ii) $\lim_{\tau \to 0^+}[D(t + \tau, x_t) - D(t, x_t)] = 0$ *uniformly for $t \in [t_0, t_0 + \delta)$,*

(iii) $K(t, x_t, x_t) - K(t, x_t, x_{t-\tau}) \to 0$ *uniformly for $t \in [t_0, t_0 + \delta)$ as $\tau \to 0^+$ and as* $\sup_{\tau+t_0 \leq s \leq t} |x(s) - x(s - \tau)| \to 0$,

(iv) *there exists a constant N such that $|K^{-1}(t, x_t, x_t)| \leq N$ for all $t \in [t_0, t_0 + \delta)$,*

(v) *there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ with $\tau(0) = 0$ and such that*

$$| L(t, x_t, x_{t-\tau}) - L^*_\beta(t, x_t, x_{t-\tau}) | \leq \tau(\beta) \sup_{-\beta \leq \theta \leq 0} | x(t + \theta) - x(t - \tau + \theta) |$$

where for a given $\beta > 0$, $\lim_{\tau \to 0^+} L^*_\beta(t, x_t, x_{t \to \tau}) = 0$ uniformly for $t \in [t_0, t_0 + \delta)$. Then for any $\omega \in F$ and any $\gamma > 0$, if $x(t)$ is a noncontinuable solution of (IVP) (2.1) – (2.2) defined on $[t_0, t_0 + \delta)$, there exists a $t^* \in [t_0, t_0 + \delta)$ such that $|x(t^*)| > \gamma$ or $x_{t^*} \notin \omega$.

Proof: By way of contradiction, if there exist a noncontinuable solution $x(t)$ of (IVP) (2.1)-(2.2) on $[t_0, t_0 + \delta)$ such that $|x(t)| \leq \gamma$ and $x_t \in \omega$ for all $t \in [t_0, t_0 + \delta)$, that is, $x \in B_\omega(t_0, \phi, \delta, \gamma)$, then first, $x(t)$ is not uniformly continuous on $[t_0, t_0 + \delta)$. Otherwise, $x(t_0 + \delta) = \lim_{t \to (t_0+\delta)^-} x(t)$ exists and thus $x_{t_0+\delta} \in \Omega$. By Theorem 3.1, $x(t)$ can be continued beyond $t_0 + \delta$.

Therefore, there exist a sufficiently small constant $\varepsilon > 0$ and sequences $\{t_k\} \subseteq [t_0, t_0 + \delta), \{\Delta_k\}$ with $\Delta_k \to 0^+$ as $k \to \infty$, and

$$| x(t_k) - x(t_k - \Delta_k) | \geq \varepsilon \tag{3.8}$$

for all $k = 1, 2, \cdots$. Now, choose a constant $N > 0$ so that

$$| K^{-1}(t, x_t, x_t) | \leq N \tag{3.9}$$

for all $t \in [t_0, t_0 + \delta)$. For there given N and $\varepsilon > 0$, by (i), (ii), (iii) and (iv), we can find constants β and $\delta_0 > 0$ so that

$$N \int_t^{t+\delta_0} m(s)ds < \frac{\varepsilon}{5} \tag{3.10}$$

$$N | K(t, x_t, x_t) - K(t, x_t, x_{t-\tau}) | < \frac{1}{5} \tag{3.11}$$

if $\sup_{\tau+t_0\le s\le t} | x(s) - x(s-\tau) | \le \varepsilon$, and

$$N \mid D(t+\tau, x_t) - D(t, x_t) \mid < \frac{\varepsilon}{5} \tag{3.12}$$

$$N\tau(\beta) < \frac{1}{5}, \beta < \frac{\delta}{2} \tag{3.13}$$

$$N \mid L^*_\beta(t, x_t, x_{t-\tau}) \mid < \frac{\varepsilon}{5} \tag{3.14}$$

for all $t \in [t_0, t_0+\delta)$ and $0 < \tau < \delta_0$. Now define a sequence $\{s_k\}$ in the following pattern

$$s_k = \inf\{t \in (t_0+\delta-\beta, t_0+\delta); | x(t) - x(t-\Delta_k) | \ge \varepsilon\} \tag{3.15}$$

Then

$$| x(s_k) - x(s_k - \Delta_k) | = \varepsilon.$$

Now, find a constant $N_1 > 0$ so that for all $k \ge N_1$, we have

$$\Delta_k < \delta_0 \tag{3.16}$$

and thus for all $k \ge N_1$, we have

$$N \int_{s_k-\Delta_k}^{s_k} m(s)ds < \frac{\varepsilon}{5} \tag{3.17}$$

$$N \mid K(s_k, x_{s_k}, x_{s_k}) - K(s_k, x_{s_k}, x_{s_k-\Delta_k}) \mid < \frac{1}{5} \tag{3.18}$$

$$N \mid D(s_k, x_{s_k-\Delta_k}) - D(s_k - \Delta_k, x_{s_k-\Delta_k}) \mid < \frac{\varepsilon}{5} \tag{3.20}$$

$$\begin{aligned} &N \mid L(s_k, x_{s_k}, x_{s_k-\Delta_k}) - B^*_\beta(s_k, x_{s_k}, x_{s_k-\Delta_k}) \mid \\ \le\ &\frac{1}{5} \sup_{-\beta\le\theta\le 0} | x(s_k+\theta) - x(s-\Delta_k+\theta) | \\ \le\ &\frac{\varepsilon}{5} \end{aligned} \tag{3.21}$$

On the other hand, we have

$$\begin{aligned} &D(s_k, x_{s_k}) - D(s_k - \Delta_k, x_{s_k-\Delta_k}) \\ =\ &D(s_k, x_{s_k}) - D(s_k, x_{s_k-\Delta_k}) + D(s_k, x_{s_k-\Delta_k}) - D(s_k - \Delta_k, x_{s_k-\Delta_k}) \\ =\ &[K(s_k, x_{s_k}, x_{s_k-\Delta_k}) - K(s_k, x_{s_k}, x_{s_k})][x(s_k) - x(s_k - \Delta_k)] \\ &+ K(s_k, x_{s_k}, x_{s_k})[x(s_k) - x(s_k - \Delta_k)] + L(s_k, x_{s_k}, x_{s_k-\Delta_k}) - L^*_\beta(s_k, x_{s_k}, x_{s_k-\Delta_k}) \\ &+ L^*_\beta(s_k, x_{s_k}, x_{s_k-\Delta_k}) + D(s_k, x_{s_k-\Delta_k}) - D(s_k - \Delta_k, x_{s_k-\Delta_k}) \end{aligned}$$

Therefore

$$
\begin{aligned}
&| x(s_k) - x(s_k - \delta_k) | \\
\leq\ & | K^{-1}(s_k, x_{s_k}, x_{s_k}) | \Big[\int_{s_k-\Delta_k}^{s_k} m(s)ds \\
&+ | K(s_k, x_{s_k}, x_{s_k-\Delta_k}) - K(s_k, x_{s_k}, x_{s_k}) | | x(s_k) - x(s_k - \Delta_k) | \\
&+ | L(s_k, x_{s_k}, x_{s_k-\Delta_k}) - L^*_\beta(s_k, x_{s_k}, x_{s_k-\Delta_k}) | \\
&+ | L^*_\beta(s_k, x_{s_k}, x_{s_k-\Delta_k}) | + | D(s_k, x_{s_k-\Delta_k}) - D(s_k - \Delta_k, x_{s_k-\Delta_k}) | \Big] \\
<\ & \varepsilon
\end{aligned}
$$

This is contrary to $|x(s_k) - x(s_k - \Delta_k)| = \varepsilon$. The proof is completed.

In the following, for any $(t_0, \phi) \in [0, \infty) \times \Omega$, and any constants $\varepsilon, \delta, \gamma > 0$, $C_\varepsilon(t_0, \phi, \delta, \gamma)$ denotes the set of all functions $x : (-\infty, t_0 + \delta] \to R^n$ so that $\|x_{t_0} - \phi\|_B < \varepsilon$, $x : [t_0, t_0 + \delta] \to R^n$ is continuous and $|x(t) - \phi(0)| \leq \gamma$.

Theorem 3.4(Continuous Dependence): *Suppose that for any $(t_0, \phi) \in [0, \infty) \times \Omega$, the solution of $(IVP)\,(2.1)-(2.1)$ is unique. Besides, suppose that for any $C_\varepsilon(t_0, \phi, \delta, \gamma)$, we have*

(i) *there exists a locally integrable function $m : [t_0, t_0 + \delta) \to [0, \infty)$ such that for any $x \in C_\varepsilon(t_0, \phi, \delta, \gamma)$, $f(t, x_t)$ is measurable and $|f(t, x_t)| \leq m(t)$,*

(ii) $\lim_{\tau \to 0^+}[D(t + \tau, x_t) - D(t, x_t)] = 0$ *uniformly for $x \in C_\varepsilon(t_0, \phi, \delta, \gamma)$ and $t \in [t_0, t_0 + \delta]$,*

(iii) $K(t, x_t, x_t) - k(t, x_t, x_{t-\tau}) \to 0$ *uniformly for $t \in [t_0, t_0 + \delta]$ and for $x \in C_\varepsilon(t_0, \phi, \delta, \gamma)$ as $\tau \to 0^+$ and $\sup_{\tau + t_0 \leq \theta \leq t} |x(\theta) - x(\theta - \tau)| \to 0$,*

(iv) *there exists a constant N such that $|K^{-1}(t, x_t, y_t)| \leq N$ for all $t \in [t_0, t_0 + \delta]$ and for all $x, y \in C_\varepsilon(t_0, \phi, \delta, \gamma)$,*

(v) *there exists a continuous function $\tau : [0, \infty) \to [0, \infty)$ with $\tau(0) = 0$ such that*

$$| L(t, x_t, x_{t-\tau}) - L^*_\beta(t, x_t, x_{t-\tau}) | \leq \tau(\beta) \sup_{-\beta \leq s \leq 0} | x(t + s) - x(t - \tau + s) |$$

where for a given $\beta > 0$, $\lim_{\tau \to 0^+} L^*_\beta(t, x_t, x_{t-\tau}) = 0$ uniformly for $t \in [t_0, t_0 + \delta]$ and for $x \in C_\varepsilon(t_0, \phi, \delta, \gamma)$,

(vi) *for any $x, y \in C_\varepsilon(t_0, \phi, \delta, \gamma)$, if $\|x_{t_0} - y_{t_0}\| \to 0$ and $\sup_{t_0 \leq s \leq t_0 + \delta} |x(s) - y(s)| \to 0$, then $D(t, x_t) \to D(t, y_t)$ and $\int_{t_0}^t [f(s, x_s) - f(s, y_s)]ds \to 0$.*

If x is a noncontinuable solution of $(IVP)\,(2.1) - (2.2)$ defined on $[t_0, t_0 + b)$, then for any $\varepsilon > 0$ and $\delta \in (0, b)$, we can find a $\sigma > c$ so that if $\|\phi - \psi\|_B < \sigma$, then $|x(t) - y(t)| < \varepsilon$ for $t \in [t_0, t_0 + \delta]$, where $y(t)$ is a solution of (2.1) through (t_0, ψ).

Proof: By way of contradiction, if the conclusion above is not true, then there exist $\varepsilon > 0$, sequences $\{t_k\} \subseteq [t_0, t_0 + \delta]$ and $\{\phi_k\} \subseteq \Omega$ such that

$$
\begin{aligned}
&\| \phi_k - \phi \| < \frac{1}{k} \\
&| y_k(t_k) - x(t_k) | = \varepsilon \qquad (3.22)
\end{aligned}
$$

and

$$| y_k(t) - x(t) | < \varepsilon \text{ for } t \in [t_0, t_k),$$

where $y_k(t)$ is a solution of the following IVP

$$\frac{d}{dt} D(t, y_t) = f(t, y_t),\ y_{t_0} = \phi_k \tag{3.23}$$

Without loss of generality, we may assume $t_k \to \bar{t} \in [t_0, t_0 + \delta]$. Now define a function sequence. $\{z_k\}$ as following

$$z_k(t) = \begin{cases} y_k(t) & \text{for} \quad t \in [t_0, t_k] \\ y_k(t_k) & \text{for} \quad t \in [t_k, \bar{t}] \text{ if } t_k < \bar{t} \end{cases}$$

Using the same argument as that of Theorem 3.3, we can assume that $\{z_k\}$ is equicontinuous in $t \in [t_0, \bar{t}]$. By Ascoli-Arzela theorem, without loss of generality, we can find a continuous $y : (-\infty, \bar{t}] \to R^n$ so that $lim_{k\to\infty} \sup_{t_0 \le s \le \bar{t}} | z_k(s) - y(\dot{s}) | = 0$ and $y(s) = \phi(s)$ for $s \le t_0$.

Now considering the equation (3.23), we get

$$D(t, y_{kt}) - D(t_0, \phi_k) = \int_{t_0}^{t} f(s, y_{ks}) ds$$

By (vi) and Lebesgue dominate convergence theorem, we obtain

$$D(t, y_t) - D(t_0, \phi) = \int_{t_0}^{t} f(s, y_s) ds$$

This means that $y(t) = x(t)$ for $t \in [t_0, \bar{t}]$ by the uniqueness assumption of the solution of (IVP)(2.1)-(2.2). This is contrary to (3.22) and

$$\lim_{k\to\infty} \sup_{t_0 \le s \le \bar{t}} | z_k(s) - y(s) | = 0.$$

The proof is completed.

4. Implication to Admissible Phase Spaces

In this section, we derive some consequences of the general theory of the previous section to admissible phase spaces, for the sake of convenience of application to Volterral integrodifferential equations.

Definition 4.1: A space B defined in Section 2 is said to be an admissible phase space if there exist a constant $J > 0$ and continuous functions $K, M : [0, \infty) \to [0, \infty)$ such that the following conditions hold

Let $0 \leq a < A$. If $x : (-\infty, A) \to R^n$ is defined on $(-\infty, A)$ with $x_a \in B$ and $x : [a, A) \to R^n$ being continuous, then for all $t \in [a, A)$,

(B1) $x_t \in B$,

(B2) $t \in [a, A) \to x_t \in B$ is continuous with respect to $|| \cdot ||_B$,

(B3) $||x_t||_B \leq K(t-a) \max_{a \leq s \leq t} |x(s)| + M(t-a)||x_a||_B$,

(B4) $|\phi(0)| \leq J||\phi||_B$ for all $\phi \in B$.

For examples of admissible phase spaces, we refer to [15] and [19]

It is easy to prove the following statement

Lemma 4.1: *Let B be an admissible phase space, $D : [0, \infty) \times B \to R^n$ is continuous and*

$$D(t, \phi) - D(t, \psi) = K(t, \phi, \psi)[\phi(0) - \psi(0)] + (t, \phi, \psi)$$

where $(t, \phi, \psi) \in [0, \infty) \times B \times B, K : [0, \infty) \times B \times B \to R^{n \times n}, L(t, \phi, \psi) : [0, \infty) \times B \times B \to R^n$ are continuous and satisfy the following conditions

(i) $\det K(t, \phi, \phi) \neq 0$ *for all* $(t, \phi) \in [0, \infty) \times B$,

(ii) *for any $(t_0, \phi) \in [0, \infty) \times \Omega$, there exist constants $\delta, \gamma > 0$ and $k_1 \in [0, 1)$ such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$, we have*

$$| K^{-1}(t_0, \phi, \phi)L(t, x_t, y_t) | \leq k_1 \sup_{t_0 \leq s \leq t} | x(s) - y(s) |$$

Then D is generalized atomic on B.

Theorem 4.1: *Suppose B is an admissible phase space, $D : [0, \infty) \times B \to R^n$ and $f : [0, \infty) \times B, K : [0, \infty) \times B \times B \to R^n$ are continuous functionals, and D is generalized atomic on B. Then $(IVP)(2.1) - (2.2)$ has a solution.*

Theorem 4.2: *Suppose that B is an admissible phase space, $D, f : [0, \infty) \to R^n$ are continuous, D is generalized atomic on B. Moreover, suppose that for any $(t_0, \phi) \in [0, \infty) \times B$, there exist $\delta, \gamma > 0$ and a function $g : [t_0, t_0 + \delta] \to [0, \infty)$ continuous at $t = t_0, g(t_0) = 0$, and such that for any $x, y \in A(t_0, \phi, \delta, \gamma)$, we have*

$$| \int_{t_0}^{t} [f(s, x_s) - f(s, y_s)] ds | \leq g(t) \sup_{t_0 \leq s \leq t} | x(s) - y(s) |$$

Then $(IVP)(2.1) - (2.2)$ has a unique solution.

Theorem 4.3: *Suppose that all conditions of Theorem 4.1 hold, and*

(i) *$f(t, \phi)$ and $K^{-1}(t, \phi, \phi)$ are completely continuous,*

(ii) *D and K are uniformly continuous on any bounded set of $[0, \infty) \times B$, or $[0, \infty) \times B \times B$, respetively,*

(iii) *for any bounded closed set W of $[0, \infty) \times B$, there exist a contionuous function $\tau : [0, \infty) \to [0, \infty), \tau(0) = 0$, and a locally bounded function $N : [0, \infty) \to [0, \infty)$ such* *that for any* $x, y : (-\infty, t_0+\delta) \to R^n (\delta > 0)$, *if* $(t, x_t), (t, y_t) \in W$ *for* $t \in [t_0, t_0+\delta)$, *then*

$$|L(t, x_t, y_t)| \leq \tau(\beta) \sup_{-\beta \leq s \leq 0} |x(t+s) - y(t+s)| + N(\beta)||x_{t-\beta} - y_{t-\beta}||_B \text{ for } \beta \in [0, \delta).$$

Then for any noncontinuable solution $x(t)$ *of* $(IVP)(2.1)-(2.2)$ *on* $[t_0, t_0+\delta)(\delta < \infty)$, *there exists a sequence* $t_n \to (t_0+\delta)^-$ *such that* $\lim_{n\to\infty} |x(t_n)| = \infty$.

Theorem 4.4: *Suppose that all conditions of Theorem 4.2 and Theorem 4.3 hold, then the solution* $x(t; t_0, \phi)$ *of* $(IVP)(2.1)-(2.2)$ *depends on its initial value* ϕ *continuously.*

A typical example of admissible phase spaces is the subspace $UC_g = \{UC_g, |\cdot|_g\}$ *of* C_g, *defined by*

$$UC_g = \{\phi \in C_g; \frac{\phi}{g} \text{ is uniformly continuous on } (-\infty, 0]\}$$

where

$$C_g = \{\phi : (-\infty, 0] \to R^n; | \phi |_g = \sup_{s \leq 0} | \phi(s) | / g(s) < \infty\}$$

and $g : (-\infty, 0] \to [1, \infty)$ *satisfies the following condition*

(g1) $g : (-\infty, 0] \to [1, \infty)$ *is a continuous nonincreasing function on* $(-\infty, 0]$ *such that* $g(0) = 1$,

(g2) $\dfrac{g(s+u)}{g(s)} \to 1$ *uniformly on* $(-\infty, 0]$ *as* $u \to 0^-$

(see, eg.[12]). In the following part, we will find a suitable function g *so that* C_g *constitutes an admissible phase space for the following neutral Volterra integrodifferential equation*

$$\frac{d}{dt}[x(t) - \sum_{i=1}^{\infty} B_i(t, x(t-r_i)) - E(t, x_t |_{[-r,0]}) - \int_{-\infty}^{t} G(t, t+s, x(s))ds]$$
$$= \sum_{i=1}^{\infty} A_i(t, x(t-r_i)) + F(t, x_t |_{[-r,0]}) + \int_{-\infty}^{t} H(t, t+s, x(s))ds \qquad (4.1)$$

in which, for any $r \geq 0$, C_r *is defined by*

$$C_r = C([-r, 0], R^n)$$

with the norm $||\phi||_r = \max_{-r \leq s \leq 0} |\phi(s)|, E, F, : [0, \infty) \times C_r \to R^n, B_i, A_i : [0, \infty) \times R^n \to R^n, G, H : [0, \infty) \times R \times R^n \to R^n$ *are continuous.*

Definition 4.1: Let $B_i : [0, \infty) \times R^n \to R^n$ be a function sequence. $\{B_i\}$ is said to have discrete fading memory if, for each $\epsilon > 0$ and each $B > 0$, there exists an integer K so that

$$\sum_{i=K}^{\infty} | B_i(t, x_i) | < \epsilon$$

for all $t \geq 0$ and all $x_i \in R^n$ with $|x_i| \leq B$.

Definition 4.2: [2] Let $G : [0, \infty) \times R \times R^n$. G is said to have integral fading memory if, for each $\epsilon > 0$ and each $B > 0$, there exists $K > 0$ such that

$$\int_{-\infty}^{-K} | G(t, t+s, \phi(s)) | \, ds \; < \; \epsilon$$

for all $t \geq 0$ and $\phi \in BC$ with $||\phi||_\infty \leq B$, where $BC = C_g$ and $||\phi||_\infty = |\phi|_g$ with $g(s) = 1$ for all $s \in (-\infty, 0]$.

Lemma 4.2: *Suppose that $G : [0, \infty) \times R \times R^n \to R^n$ is continuous and has integral fading memory, $B_i : [0, \infty) \times R^n \to R^n$ is continuous and $\{B_i\}$ has discrete fading memory. Then for each $\alpha > 0, r \geq 0$ and each unbounded increasing positive real number sequence $\{r_i\}$, there exists a function $g : (-\infty, 0] \to [1, \infty)$ satisfying $(g1), (g2)$ and (g3) $g(s) \to \infty$ as $s \to -\infty$ with $g(s) = 1$ on $[-r, 0]$ such that*

$$\sum_{i=[K]}^{\infty} | Q_i(t, \phi(-r_i)) | + \int_{-\infty}^{-K} | Q(t, t+s, \phi(s)) | \, ds \to 0 \text{ as } K \to \infty$$

uniformly for $t \geq 0$ and $\phi|_g \leq \alpha$, where $[K]$ denotes the integer part of K.

Proof: For each $i \geq 1$, take $\epsilon_i = \dfrac{1}{i^2}$ and find n_i such that

$$\int_{-\infty}^{-r_{n_i}} | G(t, t+s, \phi(s)) | \, ds \; < \; \frac{\epsilon_i}{2}$$

and

$$\sum_{k=n_i}^{\infty} | B_k(t, x_k) | < \frac{\epsilon_i}{2}$$

for all $t \geq 0, ||\phi||_\infty \leq \alpha(i+2)$ and $|x_k| \leq \alpha(i+2)$ for $k \geq n_i$. Obviously, we can choose the sequence $\{n_i\}$ so that $r_{n_1} \geq r, r_{ni+1} \geq r_{n_i} + 1$ and $r_{n_i} \to \infty$ as $i \to \infty$.

Now, define a continuous, piece linear function $g : (-\infty, 0] \to [1, \infty)$ by

(i) $g(s) = 1$ on $[-r_{n_1}, 0]$,

(ii) $g(-r_{n_i}) = i + 1$ for $i > 1$,

(iii) g is linear on the interval $[-r_{n_{i+1}}, -r_{n_i}]$ for $i \geq 1$.

Let $\phi \in C_g$ with $||\phi||_{C_g} \leq \alpha$. Then $|\phi(s)| \leq \alpha g(s)$ on $(-\infty, 0]$. For each n, set

$$\phi_n(s) = \begin{cases} \phi(-r_{n_i}), \; s \in [-r_{n_i}, 0] \\ \phi(s) \; s \in [-r_{n_{i+1}}, -r_{n_i}] \\ \phi(-r_{n_{i+1}}, \; s \in (-\infty, -r_{n_{i+1}}]. \end{cases}$$

From which it follows that $|\phi_n(s)| \leq \alpha g(-r_{n_{i+1}}) \leq \alpha(i+2)$ for all $s \leq 0$. That is, $||\phi||_\infty \leq \alpha(i+2)$.

Let $l \geq j$ be chosen sufficiently large so that

$$\sum_{k=j}^{\infty} k^{-2} < \epsilon \qquad \text{and} \qquad r_{n_l} \geq n_j$$

Then for any positive integer $K > r_{n_l}$, we have

$$\begin{aligned}
&\sum_{i=k}^{\infty} \mid B_i(t,\phi(-r_i)) \mid + \int_{-\infty}^{-K} \mid G(t,s,\phi(s)) \mid ds \\
&\leq \sum_{i=n_j}^{\infty} \mid B_i(t,\phi(-r_i)) \mid + \int_{-\infty}^{-r_{n_j}} \mid G(t,s,\phi(s)) \mid ds \\
&\leq \sum_{k=j}^{\infty} \sum_{i=n_k}^{n_{k+1}} \mid B_i(t,\phi(-r_i)) \mid + \sum_{k=j}^{\infty} \int_{-r_{n_{k+1}}}^{-r_{n_k}} \mid G(t,s,\phi(s)) \mid ds \\
&\leq \sum_{k=j}^{\infty} \sum_{i=n_k}^{n_{k+1}} \mid B_i(t,\phi_k(-r_i)) \mid + \sum_{k=j}^{\infty} \int_{-r_{n_{k+1}}}^{-r_{n_k}} \mid G(t,s,\phi_k(s)) \mid ds \\
&\leq \sum_{k=j}^{\infty} \frac{\epsilon_k}{2} + \sum_{k=j}^{\infty} \frac{\epsilon_k}{2} \\
&< \epsilon
\end{aligned}$$

This completes the proof.

Lemma 4.3: *Suppose that $G : [0,\infty) \times R \times R^n \to R^n$ is continuous and has integral fading memory, $B_i : [0,\infty) \to R^n$ is continuous and $\{B_i\}$ has discrete fading memory. Then for each $\alpha > 0$, each $r \geq 0$ and each unbounded increasing positive real number sequence $\{r_i\}$, there exists a function g satisfying $(g1),(g2)$ and $(g3)$ with $g(s) = 1$ on $[-r, 0]$ such that the function $M(t,\phi)$ defined by*

$$M(t,\phi) = \sum_{i=1}^{\infty} B_i(t,\phi(-r_i)) + \int_{-\infty}^{0} G(t,t+s,\phi(s))ds$$

is continuous on $[0,\infty) \times S_g(\alpha)$.

Proof: By Lemma 4.2, we can choose a continuous function $g : (-\infty, 0] \to [1,\infty)$ satifying (g1),(g2) and (g3), and such That

$$\sum_{i=[K]}^{\infty} \mid B_i(t,\phi(-r_i)) \mid + \int_{-\infty}^{-K} \mid G(t,t+s,\phi(s)) \mid ds \to 0$$

as $K \to \infty$ uniformly for $t \geq 0$ and $|\phi|_g \leq \alpha$. Therefore for each $\epsilon > 0$ we can find an integer K so that

$$\sum_{i=[K]}^{\infty} \mid B_i(t, \phi(-r_i)) \mid + \int_{-\infty}^{-K} \mid G(t, t+s, \phi(s)) \mid ds < \frac{\epsilon}{3}$$

for all $t \geq 0$ and $|\phi|_g \leq \alpha$.

For any $\tau \in [0, \infty), B_i(t, x)(i = 1, 2, \cdots K)$ and $G(t, s, x)$ are uniformly continuous on $I_1 = [0, \tau+1] \times \{x \in R^n; |x| \leq (\alpha+1)g(-r_k)\}$ and $I_2 = [0, \tau+1] \times [-K, \tau+1] \times \{x \in R^n; |x| \leq (\alpha+1)g(-K)\}$, respectively. Therefore we can find $\delta_1 > 0$ so that if $(\tau, x), (\tau, y) \in I_1$ with $|t - \tau| < \delta_1$ and $|x - y| < \delta_1$, then

$$\mid B_i(t, x) - B_i(\tau, y) \mid < \frac{\epsilon}{6K} \text{ for } i = 1, 2 \cdots K$$

and if $(t, s, x), (\tau, u, y) \in I_2$ with $|t - \tau| < \delta_1, |s - u| < \delta_1$ and $|x - y| < \delta_1$, then

$$\mid G(t, s, x) - G(\tau, u, y) \mid \frac{\epsilon}{6K}$$

Now choose

$$\delta = \min\{1, \frac{\delta_1}{g(-r_K)}, \frac{\delta_1}{g(-r_K)}\}$$

Then if $(\tau, \psi) \in [0, \infty) \times S_g(\alpha)$, and if $|t - \tau| < \delta, |\phi - \psi|_g < \delta$, then on the interval $[-r_K, 0]$, we have

$$\begin{aligned} \mid \phi(s) - \psi(s) \mid &< \delta_g(-r_K) \ \leq \delta_1 \\ \mid \phi(s) \mid \leq [\mid \psi \mid_g + \delta]_g(-r_K) \ &\leq (\alpha+1)_g(-r_K) \end{aligned}$$

on the interval $[-r_K, 0]$, we have

$$\begin{aligned} \mid \phi(s) - \psi(s) \mid &\leq \delta_g(-K) \ \leq \delta_1 \\ \mid \phi(s) \mid \leq [\mid \psi \mid_g + \delta]_g(-K) \ &\leq (\alpha+1)_g(-K) \end{aligned}$$

and thus

$$\begin{aligned} &\mid \sum_{i=1}^{K} B_i(t, \phi(-r_i)) - \sum_{i=1}^{K} B_i(\tau, \psi(-r_i)) \mid \\ &< \sum_{i=1}^{K} \frac{\epsilon}{6K} \\ &= \frac{\epsilon}{6} \end{aligned}$$

and

$$
\begin{aligned}
&| \int_{-K}^{0} G(t,t+s,\phi(s))ds - \int_{-K}^{0} G(\tau,\tau+s,\psi(s))ds | \\
&\leq \int_{-K}^{0} | G(t,t+s,\phi(s)) - G(\tau,\tau+s,\psi(s)) | ds \\
&< \frac{\epsilon}{6}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&| \sum_{i=K}^{\infty} B_i(t,\phi(-r_i)) + \int_{-\infty}^{-K} G(t,t+s,\phi(s))ds \\
&- \sum_{i=K}^{\infty} B_i(\tau,\psi(-r_i)) - \int_{-\infty}^{-K} G(\tau,\tau+s,\psi(s))ds | \\
&\leq \sum_{i=K}^{\infty} | B_i(t,\phi(-r_i)) | + \int_{-\infty}^{-K} | G(t,t+s,\phi(s)) | ds \\
&+ \sum_{i=K}^{\infty} | B_i(\tau,\psi(-r_i)) | + \int_{-\infty}^{-K} | G(\tau,\tau+s,\psi(s)) | ds \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \frac{2\epsilon}{3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&| \sum_{i=1}^{\infty} B_i(t,\phi(-r_i)) + \int_{-\infty}^{0} G(t,t+s,\phi(s))ds \\
&- \sum_{i=1}^{\infty} B_i(\tau,\psi(-r_i)) - \int_{-\infty}^{0} G(\tau,\tau+s,\psi(s))ds | \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{6} \\
&= \epsilon
\end{aligned}
$$

This completes the proof.

For the simplicity of the following statements, we introduce the following notations. For each $M > 0$ and each $\gamma > 0$, $R^n(M,\gamma)$ and $C_r(M,\gamma)$ are defined as

$$
\begin{aligned}
R^n(M,\gamma) &= \{(x,y) \in R^n \times R^n; | x | \leq M, | y | \leq M, | x-y | \leq \gamma\} \\
C_r(M,\gamma) &= \{(\phi,\psi) \in C_r \times C_r; || \phi ||_r \leq M, || \psi ||_r \leq M, || \phi - \psi ||_r \leq \gamma\}
\end{aligned}
$$

respctively.

Lemma 4.4: *Suppose that $G : [0,\infty) \times R \times R^n \to R^n$ is continuous and has integral fading memory, $B_i : [0,\infty) \times R^n \to R^n$ is continuous and $\{B_i\}$ has discrete fading* *memory,* $E : [0, \infty) \times C_r \to R^n$ *is continuous. Besides, suppose that for each bounded closet set* $[a,b] \subseteq [0,\infty)$ *and each* $M > 0$*, there exist constants* $\gamma > 0, k \in [0,1)$ *and a continuous function* $l : [a,b] \times [a,b] \to [0,\infty)$ *such that*

$$| E(t,\phi) - E(t,\psi) | \leq k \sup_{-r \leq s \leq 0} | \phi(s) - \psi(s) |$$

for all $t \in [a,b]$ *and* $(\phi,\psi) \in C_r(M,\gamma)$*, and*

$$| G(t,s,x) - G(t,s,y) | \leq l(t,s) | x - y |$$

for all $(t,s) \in [a,b] \times [a,b]$ *and* $(x,y) \in R^n(M,\gamma)$*. Then for each* $\alpha > 0$ *and each unbounded increasing positive real number sequence* $\{r_i\}$*, there exists a function* $g : (-\infty, 0] \to [1,\infty)$ *satisfying* $(g1),(g2)$ *and* $(g3)$ *with* $g(s) = 1$ *on* $[-r,0]$ *such that the operator* $D(t,\phi)$ *defined by*

$$D(t,\phi) = \phi(0) - \sum_{k=1}^{\infty} B_k(t, \phi(-r_k)) - E(t, \phi |_{[-r,0]}) - \int_{-\infty}^{0} G(t, t+s, \phi(s))ds$$

is generalized atomic on $C_g(\alpha)$*.*

Proof: It is clear that

$$D(t,\phi) - (t,\psi) = K(t,\phi,\psi)[\phi(0) - \psi(0)] + L(t,\phi,\psi)$$

where

$$\begin{aligned} K(t,\phi,\psi) &= I(\text{the } n \times n \text{ identity matrix}), \\ L(t,\phi,\psi) &= \sum_{k=1}^{\infty} B_k(t,\psi(-r_k)) - \sum_{k=1}^{\infty} B_k(t,\phi(-r_k)) + E(t,\psi |_{[-r,0]}) - E(t,\phi |_{[-r,0]}) \\ &\quad + \int_{-\infty}^{0} G(t,t+s,\psi(s))ds - \int_{-\infty}^{0} G(t,t+s,\phi(s))ds \end{aligned}$$

Choose a function $g : (-\infty,0] \to [1,\infty)$ as in Lemma 4.2 and Lemma 4.3. Then $D(t,\phi)$ and $L(t,\phi,\psi)$ are continuous in $(t,\phi,\psi) \in [0,\infty) \times C_g(\alpha) \times C_g(\alpha)$. For any $(t_0,\phi) \in [0,\infty) \times C_g(\alpha)$, define

$$M = \alpha + 1, [a,b] = [t_0, t_0 + 1]$$

Choose corresponding

$$r \in [0,2), k \in [0,1)$$

in the assumptions, and find a sufficiently small

$$\delta \in (0, \min\{r, 1, r_1\})$$

so that

$$k+\sup_{t_0\le s\le t_0+\delta}\int_{t_0}^{t} g(t,s)ds \;=\; k_1<1$$

then, for any $(t_0,\phi)\in[0,\infty)\times C_g(\alpha), x,y:(-\infty,t_0+\delta]\to R^n$ such that $x_{t_0}=y_{t_0}=\phi, |x(t)-\phi(0)|\le\frac{\gamma}{2}, |y(t)-\phi(0)|\le\frac{\gamma}{2}$ for $t\in[t_0,t_0+\delta]$, we have

$$\begin{aligned}
&\mid x(t)-y(t)\mid\le\gamma\\
&\mid x(t)\mid\le\mid\phi(0)\mid+\frac{\gamma}{2}\le\alpha+\frac{\gamma}{2}\le\alpha+1 \;=\; M\\
&\mid y(t)\mid\le\mid\phi(0)\mid+\frac{\gamma}{2}\le\alpha+1=M
\end{aligned}$$

for $t\in[t_0,t_0+\delta]$, and thus $x(t),y(t)\in R^n(M,\gamma)$ for $t\in[t_0,t_0+\delta]$. This implies that

$$\mid G(t,s,x(s))-G(t,s,y(s))\mid\le\; g(t,s)\mid x(s)-y(s)\mid$$

for $t\in[t_0,t_0+\delta]\times[t_0,t_0+\delta]$, and thus

$$\begin{aligned}
&\mid\int_{-\infty}^{0}G(t,t+s,x_t(s))ds-\int_{-\infty}^{0}G(t,t+s,y_t(s))ds\mid\\
=&\mid\int_{-\infty}^{t}G(t,u,x(u))du-\int_{-\infty}^{t}G(t,u,y(u))du\mid\\
\le&\int_{t_0}^{t}\mid G(t,u,x(u))-G(t,u,y(u))\mid du\\
\le&\int_{t_0}^{t}G(t,u)\mid x(u)-y(u)\mid du\\
\le&\sup_{t_0\le v\le t_0+\delta}\int_{t_0}^{v}g(v,u)du\sup_{t_0\le s\le t}\mid x(s)-y(s)\mid
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mid\sum_{k=1}^{\infty}\phi_k(t,x_t(-r_k))-E(t,x_t\mid_{[-r,0]})-[\sum_{k=1}^{\infty}\phi_k(t,y_t(-r_k))-E(t,y_t\mid_{[-r,0]})]\mid\\
\le&\mid E(t,x_t\mid_{[-r,0]})-E(t,y_t\mid_{[-r,0]})\mid\\
\le&\, k\sup_{t_0\le s\le t}\mid x(s)-y(s)\mid
\end{aligned}$$

(notice that $\delta\le r_1$, and so $x_t(-r_k)=y_t(-r_k)$ for $t\in[t_0,t_0+\delta]$ and $k=1,2,\cdots$). Therefore

$$\begin{aligned}
&\mid D(t,x_t)-D(t,y_t)\mid\\
\le&[\sup_{t_0\le s\le t_0+\delta}\int_{t_0}^{s}g(s,u)du+k]\sup_{t_0\le s\le t}\mid x(s)-y(s)\mid\\
\le&\, k_1\sup_{t_0\le s\le t}\mid x(s)-y(s)\mid
\end{aligned}$$

This shows that $D(t,\phi)$ is generalized atomic on $C_g(\alpha)$. This completes the proof.

Combining Lemma 4.2,4.3 and 4.4 and by Theorem 4.1, we get following corollary.

Corollary 4.1: *Suppose that* $G, H : [0,\infty) \times R \times R^n \to R^n$ *are continuous and have integral fading memory,* $B_i, A_i : [0,\infty) \times R^n \to R^n$ *are continuous and* $\{B_i\}, \{A_i\}$ *have discrete fading memory,* $E, F : [0,\infty) \times C_r \to R^n$ *are continuous. Besides, suppose that for each bounded closet set* $[a,b] \subseteq [0,\infty)$ *and each* $M > 0$, *there exist constants* $\gamma > 0, k \in [0,1)$ *and a continuous function* $l : [a,b] \times [a,b] \to [0,\infty)$ *such that*

$$| E(t,\phi) - E(t,\psi) | \leq k \sup_{-r \leq s \leq 0} | \phi(s) - \psi(s) |$$

for all $s \in [a,b]$ *and* $(\phi,\psi) \in C_r(M,\gamma)$, *and*

$$| G(t,s,x) - G(t,s,y) | \leq l(t,s) | x - y |$$

for all $(t,s) \in [a,b] \times [a,b]$ *and* $(x,y) \in R^n(M,\gamma)$. *Then for each* $\alpha > 0$ *and each unbounded increasing positive real number sequence* $\{r_i\}$, *there exists a function g satisfying* $(g1), (g2)$ *and* $(g3)$ *with* $g(s) = 1$ *on* $[-r, 0]$ *such that for any* $\beta \in [0,\alpha), t_0 \in [0,\infty)$ *and* $\phi \in UC_g(\beta)$, *there exists a solution of the equation* (4.1) *through* (t_0, ϕ), *where* $UC_g(\beta) = \{\phi \in UC_g; |\phi|_g \leq \beta\}$.

To conclude this paper, we point out that results in papers [19]-[22] are contained in this paper. Verification is left for interested readers.

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