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NON SYMMETRIC HOMOGENEOUS BOUNDED DOMAINS IN 6 DIMENSIONAL COMPLEX SPACE

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1. Introduction

Let D be a homogeneous bounded domain in $\mathbf{C}^n$, $n = 6$. One of the problems for the homogeneous bounded domains in $\mathbf{C}^n$, $n \leq 3$ is to classify those which are not symmetric. In each $\mathbf{C}^4$ and $\mathbf{C}^5$ there is only one non-symmetric homogeneous bounded domain. In a previous paper we have given a new representation each of those non-symmetric homogeneous bounded domains in $\mathbf{C}^4$ and $\mathbf{C}^5$ ([10]).

The aim of this paper is to describe with a new representation one of the non-symmetric homogeneous bounded domains in $\mathbf{C}^n$, where $n = 6$.

The whole paper contains three paragraphs. Each of them is analyzed as follows.

In the second paragraph we give the relation between these domains, Siegel domains and normal J-algebras.

The description of this special non-symmetric homogeneous bounded domain in $\mathbf{C}^n$, $n = 6$, is included in the last paragraph.

2. Let $\mathbf{C}^n$ be the n dimensional Euclidean complex space. An open connected subset D of the $\mathbf{C}^n$ is called *domain*. We denote by $G(D)$ the group of all holomorphic automorphisms of D. If D is bounded, then $G(D)$ is a Lie group and there exists on D a volume element w which is defined by

$$w = (\sqrt{-1})^{n^2} K \, dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \ldots \wedge d\overline{z}_n$$

where $z_1, \ldots, z_n$ are complex coordinates in $\mathbf{C}^n$ and K the Bergaman function on D, which is positive. The Bergman function K gives a Kähler metric g on D defined by

$$g = \sum_{h=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^2 \log K}{\partial z_h \partial \overline{z}_\ell} dz_h \wedge d\overline{z}_\ell$$

and therefore (D, g) is a Kähler manifold. The bounded domain D in $\mathbf{C}^n$ is called *homogeneous*, if the group $G(D)$ acts transitively on D and thererore D, in this case, can be written

$$D = G(D)/H, \tag{2.1}$$

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where H is the isotropy subgroup of $G(D)$ at the point $z_0 \in \mathbf{C}^n$. The relation (2.1) can also be written as follows

$$D = G_0(D) / H_0$$

where $G_0(D)$ is the identiy component of $G(D)$ and H_0 the isotropy subgroup of $G_0(D)$ at $z_0 \in D$

It is known that there exists a solvable Lie subgroup S of $G(D)$ which can be identified with D

Therefore S is a kähler manifold on which there exists a complex structure on it denoted by J.

Let s be the Lie algebra of S which can be identified with the tangent space of S at its identity element e. The almost complex structure J on D defines an endomorphism J_0 on s with the following properties

$$J_0 : s \to s, \; J_0 : X \to J_0(X), \; J_0^2 = -id \tag{2.2}$$

This endomorphism J_0 satisfies the following relation

$$[X, Y] + J_0([J_0(X), Y]) + J_0([X, J_0(Y)]) - [J_0(X), J_0(Y)] = 0 \tag{2.3}$$

which is obtained from the fact that the almost complex structure on D is integrable. The Kähler metric g on D induces a Hermitian positive definite symmetric bilinear form B on s. From B we obtain a linear form ω defined by

$$\omega : s \to R, \; \omega : X \to \omega(X) = B(X, J_0(X)) \tag{2.4}$$

satisfying the following conditions

$$\omega([J_0(X), J_0(Y)] = \omega([X, Y]) \tag{2.5}$$

$$\omega([J_0(X), X] > 0 \qquad X \neq 0 \tag{2.6}$$

Therefore from the homogeneous bounded domain $D = G/H$ we obtain the set $\{s, J_0, \omega\}$, where s a special solvable Lie algebra, J_0 is an endomorphism on s having the properties (2.2) and (2.3) and ω linear form on s with the properties (2.5) and (2.6)

This set $\{s, J_0, \omega\}$ is called normal J-algebra

Every normal J-algebra has also the property that the operator

$$\alpha d\tau_0 : s \to s, \; \alpha d\tau_0 : \tau \to \alpha d\tau_0(\tau) = [\tau_0, \tau] \tag{2.7}$$

has only real characteristic roots $\forall \tau_0 \in s$, that is, $\alpha d\tau_0$, as a matrix, is R-triangular

The inverse is also true. Let (s, J_0, ω) be a triple, where s is a solvable Lie algebra having the property (2.7), J_0 an endomorphism on s having the properties (2.2) and (2.3) and ω a linear form on s having the properties (2.5) and (2.6). Then there exists a unique solvable Lie group S whose Lie algebra is s which can be identified with the tangent space of S at its identity e. The endomorphism J_0 on s gives arise the complex structure on S and finally the linear form ω on s induces a Hermitian inner product on s defined by

$$\langle X, Y \rangle = \omega([J_0 X, Y]) \tag{2.8}$$

which determines the Kähler metric g on S. The couple (S, g) is a Kähler manifold beholomorphically isomonphic onto homgeneous bounded domain in $\mathbf{C}^n$. In the next paragraph we shall give one triplet $(s,\ J_0,\ \omega)$ and the Kähler manifold (S, g) which is obtained by this triple.

3. We consider the solvable Lie algebra s, which can be described by the set of matrices

$$S = \left\{ A = \begin{pmatrix} 0 & X_1 & X_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_5 & X_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_9 & X_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{12} \end{pmatrix} \quad \begin{matrix} Xi\ EIR^{*} \\ L = 1, 2, \cdots, 12 \end{matrix} \right\} \tag{3.1}$$

From this constcuction of s we conclude that the endomorphism J_0 has the form

$$J_0 = (\beta_{k\ell}),\ \beta_{k\ell} \in R \qquad k = 1, \cdots, 12,\ \ell = 1, \cdots, 12 \tag{3.2}$$

which must satisfy the relations (2.2) and (2.3)

From these conditions and after a lot of estimates we obtain

$$j_0 = \begin{pmatrix} x & 0 & \frac{-1-x^2}{\lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Psi & 0 & \frac{-1-\Psi^2}{\mu} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & -x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & -\Psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z & 0 & \frac{-1-Z^2}{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pi & 0 & \frac{-1-\pi^2}{v} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K & 0 & -Z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U & 0 & -\pi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & \frac{-1-\rho^2}{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta & 0 & \frac{-1-\zeta^2}{\omega} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau & 0 & -\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & -\zeta \end{pmatrix} \tag{3.4}$$

The linear form ω, on this Lie algebra s, is difined by

$$\omega(X) = \langle X_0, X\rangle \tag{3.5}$$

where $\langle\ \rangle$ the usual inner product on s and $X_0 = (K_1, K_2, \cdots, K_{12})$ is a fixed vector. In order that ω satisfies the conditions (2.5) and (2.6) we must have

$$-K_1\lambda > 0,\ -K_2\mu > 0,\ \ldots, -K_6\omega > 0 \tag{3.6}$$

Now, we have proved the following theorem

Theorem 3.1 *There exists a homogeneous bounded domain in* C^n, $n = 6$ *having* (s, J_0, ω) *normal J-algebra, where* s, J_0 *and* ω *are given by* (3.1), (3.4) *and* (3.5) *respectively*

Now, we determine the solvable Lie group S which corresponds to the solvable Lie algebra s

We denote by $GL(s)$ the group of all nonsingular endomorphisms of s The Lie algebra $gl(s)$ of $GL(s)$ consists of all endomorphisms of s with the standard bracket operation

$$[X, Y] = XY - YX \tag{3.7}$$

The mapping

$$\alpha d : s \to gl(s), \qquad \alpha d : B \to \alpha dB \tag{3.8}$$

where

$$\alpha dB : s \to s, \qquad \alpha dB : T \to \alpha dB(T) = [T, B] \tag{3.9}$$

is a homomorphism of s onto a subalgebta $\alpha d(s)$ of $gl(s)$. Let $Int(s)$ be the analytic subgroup of $GL(s)$ whose Lie algebra is $\alpha d(s)$ which is called adjoint group of s. The group $Aut(s)$ of all automorphisms of s is a closed subgroup of $GL(s)$. Thus $Aut(s)$ has a unique analytic structure under which, it becomes a topological Lie subgroup of $GL(s)$. We denote by $d(s)$ the Lie algebra of $Aut(s)$. Now, the group $Int(s)$ is connected, so it is generated by elements $e^{\alpha dX}$, $X \in s$. Therefore $Int(s)$ is a normal subgroup of $Aut(s)$

From the above we conclude that the solvable Lie group S of s is defined

$$S = \left\{ L = \begin{pmatrix} 1 & \frac{x_1}{x_3}(e^{x_3}-1) & \frac{x_2}{x_4}(e^{x_4}-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{x_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{x_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{x_5}{x_7}(e^{x_7}-1) & \frac{x_6}{x_8}(e^{x_8}-1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{x_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{x_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x_9}{x_{11}}(e^{x_{11}}-1) & \frac{x_{10}}{x_{12}}(e^{x_{12}}-1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{x_{11}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{x_{12}} \end{pmatrix} \right\} \quad (3.10)$$

The inner product on the solvable Lie algebra is defined by

$$< X, Y > = \omega([J_0 X, Y]) \quad (3.11)$$

where ω is given by (3.5). This inner product determines the Kähler metric on S which is essentially the Bergaman metric on it

Now, we can state the following theorem

Theorem 3.2 *The homogeneous non-symmetric bounded domain in C^n $n = 6$ is biholomorphically isomorphic onto the solvable Lie group defined by* (3.10). *The Kähler metric g on S defined by the relation* (3.11).

Let F be a Lie automorphism on s. This F can be represented by matrix

$$F = \begin{pmatrix} \alpha_{11} & 0 & \alpha_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & \alpha_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{55} & 0 & \alpha_{57} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_{66} & 0 & \alpha_{68} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{99} & 0 & \alpha_{911} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{1010} & 0 & \alpha_{1012} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which becomes an isometry with respect to the inner product

$$< x, y > = < x_0, [J_0 x, y] > = \omega([Jx, y])$$

If we have

$$Fisom = \begin{pmatrix} \pm1 & 0 & \alpha_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & \alpha_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm1 & 0 & \alpha_{57} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pm1 & 0 & \alpha_{68} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & 0 & \alpha_{911} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \pm1 & 0 & \alpha_{1012} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\alpha_{13} = \frac{-yx\lambda}{1+x^2}, \quad \alpha_{24} = \frac{-2\mu y}{1+y^2}, \quad \alpha_{57} = \frac{-2kz}{1+z^2},$$
$$\alpha_{68} = \frac{-2nv}{1+n^2}, \quad \alpha_{911} = \frac{-2\tau\rho}{1+\rho^2}, \quad \alpha_{1012} = \frac{-2\omega J}{1+J^2}.$$

Therefore we have proved the following theorem.

Theorem 3.3. *The homogeneous bounded domain in C^n $n = 6$ described by the theorem 3.2 does not admit any k-symmetric structure.*

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