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COMPLETE SURFACES IN E^3 WITH CONSTANT MEAN CURVATURE

MEHMET ERDOĞAN AND TAKEHIRO ITOH

Abstract. We give a classification of surfaces in E^3 with constant mean curvature and the Gaussian curvature K not changing its sign around some point at which K vanishes.

Let M be a 2-dimensional complete oriented Riemannian surface in a Euclidean 3-space E^3 with constant mean curvature. On such surfaces Klotz and Osserman ([4]) proved that:

Any complete surface with constant mean curvature on which the Gaussian curvature does not change its sign is either a sphere, a minimal surface or a right circular cylinder.

A complete oriented pseudo-umbilical surface in E^4 with constant mean curvature($\neq 0$) and the Gaussian curvature does not change its sign was studied by the second author([3]) and he showed that such a surface is either a Clifford flat torus in 3-sphere $S^3 \subset E^4$ or a sphere in E^3.

In the present paper, the result obtained by Klotz and Osserman will be proved under a weaker condition. In deed we will prove the following

Theorem.. *Let M be a complete, connected and orientable surface in E^3 with constant mean curvature. If the Gaussian curvature K does not change its sign on a neighborhood of some point at which K vanishes, then M is either a minimal surface, a sphere or a right circular cylinder.*

1. Preliminaries

Let M be a 2-dimensional complete oriented Riemannian manifold isometrically immersed in Euclidean 3-space E^3 with the induced Riemannian structure through the immersion $\iota : M \to E^3$. Let $F(M)$ and $F(E^3)$ be the bundles of all orthonormal frames over M and E^3, respectively. Let B be the set of all elements (p, e_1, e_2, e_3) such that $(p, e_1, e_2) \in F(M)$ and $(p, e_1, e_2, e_3) \in F(E^3)$ whose orientation is coherent with the one

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of E^3 identifying $p \in M$ and $\iota(p)$ in E^3 and e_i with $d\iota(e_i)$ for $i = 1, 2$. Then B is naturally considered as a differentiable submanifold of $F(E^3)$.

Let ω_i and ω_{ij}, ω_{i3} be dual and connection forms with respect to the above orthonormal frames associated with the immersion. Then, as is well known, we have

$$\begin{cases} dp &= \sum_i \omega_i e_i, \\ de_i &= \omega_{ij} e_j + \omega_{i3} \quad i \neq j, \\ de_3 &= \sum_i \omega_{3i} e_i, \end{cases} \tag{1.1}$$

$$\begin{cases} d\omega_i &= \omega_{ij} \wedge \omega_j, \\ d\omega_{12} &= \omega_{13} \wedge \omega_{32}, \\ d\omega_{i3} &= \omega_{ij} \wedge \omega_{j3}, \quad i \neq j, \end{cases} \tag{1.2}$$

$$\omega_{i3} = -\omega_{3i} = \sum_j h_{ij} \omega_j, \qquad h_{ij} = h_{ji}, \tag{1.3}$$

where and throughout this paper we use the convention on indices: $i, j = 1, 2$. A matrix $H_3 = (h_{ij})_{1 \leq i,j \leq 2}$ may be called *the second fundamental form* of M. The *mean curvature* H is given by $H = (1/2) \sum_{i=1}^{2} h_{ii}$. M is called to be *umbilic* at point $p \in M$ if $h_{11} = h_{22}$ and $h_{12} = 0$ at p and to be *totally umbilic* if M is umbilic everywhere. We call M is *minimal* at a point $p \in M$ if the mean curvature vanishes at p.

We may consider M as a Riemannian surface by Chern([2]), because M is a 2-dimensional oriented Riemannian manifold. Then we can choose isothermal coordinates (u, v) on a neighborhood of a point on M.

The Gaussian curvature K is given by the equation $d\omega_{12} = -K\omega_1 \wedge \omega_2$. On the other hand the equation (1,2) and (1,3) imply that $d\omega_{12} = -detH_3\omega_1 \wedge \omega_2$. Then we have

$$K = H^2 - (\lambda^2 + \mu^2), \tag{1.4}$$

where we write H_3 by $h_{11} = H + \lambda$, $h_{22} = H - \lambda$ and $h_{12} = h_{21} = \mu$.

2. Proof of Theorem

Let M be a 2-dimensional complete oriented Riemannian manifold isometrically immersed in Euclidean 3-space E^3 with the induced Riemannian structure through the immersion $\iota : M \to E^3$. We first prove the following well known

Lemma 1. *If the mean curvature H of M is constant, then either M is totally umbilic or the set of all umbilic points of M is an isolated set of M.*

Proof. We may consider isothermal coordinates (u, v) and a frame field $(p, e_1, e_2, e_3) \in B$ on a neighborhood U of a point $p \in M$ such that

$$ds^2 = E\{du^2 + dv^2\}, \qquad \omega_1 = \sqrt{E}du, \qquad \omega_2 = \sqrt{E}dv. \tag{2.1}$$

In this case we may write $h_{11} = H + \lambda$, $h_{22} = H - \lambda$ and $h_{12} = h_{21} = \mu$. Then, using (1.2), (1.3) and (2.1), we easily see that the complex valued function $\omega := E\lambda - iE\mu$ is complex analytic with respect to $z = u + iv$ in U. By this fact we can show our assertion.

q.e.d. Now, we will give the proof of Theorem. If the mean curvature $H = 0$, then M is a minimal surface. Hence, we may consider only the case: $H \neq 0$.

If there are no points at which the Gaussian curvature K vanishes, then K does not change its sign on M, so such surfaces are classified by Klotz and Osserman([4]). Therefore, we may consider the case:

$H \neq 0$ and neither $K \geq 0$ nor $K \leq 0$ holds identically on M.

In this case our assumption means that *the Gaussian curvature K does not change its sign on a neighborhood $U = U(p_0)$ of some point p_0 at which K vanishes.* Taking account of Lemma 1 and $H \neq 0$, we may moreover assume that M is umbilic free on U. Then, we have a neighborhood V of p_0 in U where we can choose a frame field $(p, e_1, e_2, e_3) \in B$ such that

$$\omega_{13} = (H + \lambda)\omega_1 \qquad \text{and} \qquad \omega_{23} = (H - \lambda)\omega_2, \tag{2.2}$$

where $\lambda = \lambda(u, v)$ is a positive differentiable function on V. Taking exterior derivative of the equations (2.2) and using the structure equations (1.2), we have

$$\begin{cases} d\omega_{13} = d\lambda \wedge \omega_1 + (H + \lambda)d\omega_1 = (H - \lambda)d\omega_1, \\ d\omega_{23} = -d\lambda \wedge \omega_2 + (H - \lambda)d\omega_2 = (H + \lambda)d\omega_2, \end{cases}$$

which imply that

$$d(\sqrt{\lambda}\omega_i) = 0, \qquad \text{for any } i = 1, 2.$$

Hence there exists a neighborhood W of p_0 in V where we can choose isothermal coordinates (u, v) satisfying

$$ds^2 = \frac{1}{\lambda}\{du^2 + dv^2\}, \qquad \omega_1 = \frac{1}{\sqrt{\lambda}}du, \qquad \omega_2 = \frac{1}{\sqrt{\lambda}}dv.$$

As is well known, with respect to the isothermal coordinates the Gaussian curvature K is given by

$$K = \frac{\lambda}{2}\Delta \log \lambda. \tag{2.3}$$

On the other hand, as is stated in Preliminaries,

$$K = \det H_3 = H^2 - \lambda^2. \tag{2.4}$$

Then $K \leq 0$ on U implies, together with (2.3) and (2.4), the function $\log \lambda$ is a superharmonic function on W and takes its minimum at p_0. Similarly, $K \geq 0$ on U implies that $\log \lambda$ is a subharmonic function on W and takes its maximum at p_0. Hence, in both cases $\log \lambda$ must be constant on W. That is, the Gaussian curvature K vanishes on an open subset of M, which implies that $K = 0$ identically on M, because a surface with constant mean curvature is real analytic. Hence, in this case M must be a right circular cylinder.

Thus we have proved the theorem.

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Department of Mathematics Firat University Elaziğ Turkey, Institute of Mathematics University of Tsukuba Tsukuba-shi 305 Ibaraki Japan.