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ON SOME APPROXIMATION PROBLEMS IN METRIC SPACES

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In this paper we consider the problem of simultaneous characterization of a set of elements of best approximation and characterization of elements of ε-approximation in metric spaces.

1. Introduction

The theory of best approximation in metric spaces is comparatively less developed than that in normed linear spaces or linear metric spaces. Only a few mathematicians have pursued this study in sporadic attempts. The author in a series of papers has also made an attempt in this direction and the present paper is also a step in the same direction.

In this paper we discuss the problem of simultaneous characterization of a set of elements of best approximation from which follows a characterization of semi-Chebyshev subspaces and also give a characterization of elements of ε-approximation in metric spaces.

2. Simultaneous characterization of set of elements of best approximation

The problem of characterization of elements of best approximation in metric spaces was considered by Mustăta [3]. A natural generalization of this is the following problem of simultaneous characterization of a set of elements of best approximation:

Given a non-empty subset Y of a metric space $(X,d), x \in X\backslash Y$ and a subset M of Y, what are the necessary and sufficient conditions in order that every element $y_0 \in M$ be an element of best approximation to x by the elements of Y?

We shall answer this in Theorems 2.2 and 2.3. As a consequence of Theorem 2.2, we get a characterization of semi-Chebyshev subspaces (Corollary 2.1).

Let (X,d) be a metric space and x_o a fixed point of X. The set

$$X_o^{\#} = \{f : X \longrightarrow \mathbb{R}, \sup_{x,y \in X\, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)} < \infty, f(x_0)=0\},$$

of Lipschitz functions on X vanishing at x_o, is a Banach space (even a conjugate Banach space-see Johnson [2]) with the usual operations of addition, multiplication by real scalars

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$$\| f \|_X = \sup_{x,y \in X\, x \neq y} \frac{| f(x) - f(y) |}{d(x,y)}, \; f \in X_o^{\#}.$$

If Y is a subset of a metric space (X, d) and $x \in X$ then an element $y_o \in Y$ is said to be *an element of best approximation of x by elements of Y* if $d(x, y_o) = d(x, Y)$. We shall denote by $L_Y(x)$ the set of all best approximations to x in Y. The set Y is said to be *semi-Chebyshev* if $L_Y(x) = \phi$ or singleton for each $x \in X$.

Mustăta [3] gave the following characterization of elements of best approximation in metric spaces. A similar characterization in normed linear spaces was obtained by Singer (see [7]) and in linear metric spaces by Pantelidis [6].

Theorem 2.1. *Let Y be a subset of a metric space (X, d) such that $x_o \in Y$ and let $x \in X \backslash Y$. Then $y_o \in Y$ is an element of best approximation for x by elements of Y if and only if there is an $f \in X_o^{\#}$ such that*
(i) $\|f\|_X = 1$
(ii) $f|_Y = 0$
(iii) $|f(x) - f(y_o)| = d(x, y_o)$.

The following theorem gives simultaneous characterization of a set of elements of best approximation in metric spaces. In normed linear spaces a similar characterization was given by Singer (see [7]) and in linear metric spaces by Narang and Khanna [5].

Theorem 2.2. *Let Y be a subset of a metric space (X, d) such that $x_o \in Y$ and let $x \in X \backslash Y$ and $M \subset Y$. Then $M \subset L_Y(x)$ if and only if there exists an $f \in X_o^{\#}$ such that*
(a) $\|f\|_X = 1$
(b) $f|_Y = 0$
(c) $|f(x) - f(y)| = d(x, y)$ for all $y \in M$.

Proof. Suppose that $M \subset L_Y(x)$ and $y_o \in M$. Then $y_o \in L_Y(x)$ and so by Theorem 2.1, there exists an $f \in X_o^{\#}$ satisfying (a),(b) and $|f(x) - f(y_o)| = d(x, y_o)$.

Now, let $y \in M$. Then $y \in L_Y(x)$ i.e. $d(x, y) = d(x, Y) = d(x, y_o)$. Consider

$$\begin{aligned} | f(x) - f(y) | &= | f(x) - f(y_o) | \text{ as } f |_Y = 0 \\ &= d(x, y_o) \\ &= d(x, y). \end{aligned}$$

Thus (c) is also satisfied.

Conversely, suppose that there exists an $f \in X_o^{\#}$ satisfying (a),(b) and (c) and let $y_o \in M$. Then by Theorem 2.1, $y_o \in L_Y(x)$. Hence $M \subset L_Y(x)$.

As a consequence of Theorem 2.2, we get the following characterization of semi-Chebyshev subspaces of metric spaces.

Corollary 2.1. Let Y be a subset of a metric space (X, d) such that $x_o \in Y$. Then the following statements are equivalent:

(i) Y is semi-Chebyshev

(ii) There exist no $f \in X_o^{\#}$, $x_1 \in X$ and $y_1, y_2 \in Y, y_1 \neq y_2$ such that

(a) $\|f\|_X = 1$

(b) $f|_Y = 0$

(c) $f(x_1) = d(x_1, y_1) = d(x_1, y_2)$.

An independent proof of this result was given by Mustăta [4].

Let Y is a subset of a metric space (X, d) and $x_o \in Y$. If

$$Y^{\perp} = \{f \in X_o^{\#} : f|_Y = 0\},$$

and

$$d_{Y^{\perp}}(x, y) = \sup_{f \in Y^{\perp} \setminus \{0\}} \frac{|f(x) - f(y)|}{\|f\|_X}, \ x, y \in X,$$

we have the following inequality (see [3]):

$$d_{Y^{\perp}}(x, y) \leq d(x, y) \text{ for all } x, y \in X.$$

The following characterization of elements of best approximation was given by Mustăta [3]:

Theorem 2.3. *Let Y be a subset of a metric space (X, d) such that $x_o \in Y$ and let $x \in X \setminus Y$. Then $y_o \in Y$ is an element of best approximation for x by elements of Y if and only if*

$$d_{Y^{\perp}}(x, y_o) = d(x, y_o).$$

Using Theorem 2.2, we get another result on the simultaneous characterization of a set of elements of best approximation in metric spaces from which Theorem 2.3 follows as a corollary.

Theorem 2.4. *Let Y be a subset of a metric space (X, d) such that $x_o \in Y$ and let $x \in X \setminus Y$ and $M \subset Y$. Then $M \subset L_Y(x)$ if and only if*

$$d_{Y^{\perp}}(x, y) = d(x, y)$$

for all $y \in M$.

Proof. Let $M \subset L_Y(x)$. Then by Theorem 2.2 there exists an element $f \in Y^{\perp}$ such that $\|f\|_X = 1$ and $|f(x) - f(y)| = d(x, y)$ for all $y \in M$. We have

$$d_{Y^{\perp}}(x, y) = \sup_{g \in Y^{\perp} \setminus \{0\}} \frac{|g(x) - g(y)|}{\|g\|_X} \geq \frac{|f(x) - f(y)|}{\|f\|_X} = d(x, y)$$

and as $d_{Y^{\perp}}(x, y) \leq d(x, y)$ for all $x, y \in X$, we have $d_{Y^{\perp}}(x, y) = d(x, y)$ for all $y \in M$.

Conversely, suppose $d_{Y^\perp}(x,y) = d(x,y)$ for all $y \in M$. Then for any $y_0 \in M$ and $z \in Y$ we have

$$\begin{aligned} d(x,y_0) &= d_{Y^\perp}(x,y_0) \\ &= \sup_{f \in Y^\perp \setminus \{0\}} \frac{| f(x) - f(y_0) |}{||f||_X} \\ &= \sup_{f \in Y^\perp \setminus \{0\}} \frac{| f(x) - f(z) |}{||f||_X} \\ &= d_{Y^\perp}(x,z) \\ &= d(x,z) \end{aligned}$$

Hence $y_0 \in L_Y(x)$ i.e. $M \subset L_Y(x)$.

3. A characterization of elements of ε-approximation

Let Y be a subset of a metric space (X,d), $x \in X$ and $\varepsilon > 0$. An element $y_0 \in Y$ is said to be *an element of ε-approximation of x by elements of Y* if $d(x,y_0) \leq d(x,Y) + \varepsilon$. We shall denote by $L_Y(x,\varepsilon)$, the set of all elements of ε-approximation to x. In particular, for $\varepsilon = 0$ we find again the elements of best approximation of x and respectively the set $L_Y(x)$. The concept of ε-approximation was introduced by R.C. Buck [1] who used the term 'elements of good approximation'. Buck [1] and Singer (see [7]) characterized elements of ε-approximation in normed linear spaces. In linear metric spaces these elements were characterized by Narang and Khanna [5]. Here we give a characterization of such elements in metric spaces (Theorem 3.1). In the particular case when $\varepsilon = 0$, Theorem 3.1 reduces to Theorem 2.1 on the characterization of elements of best approximation.

Theorem 3.1. Let Y be a subset of a metric space (X,d) such that $x_0 \in Y$ and let $x \in X \setminus Y$, $y_0 \in Y$ and $\varepsilon > 0$. Then $y_0 \in L_Y(x,\varepsilon)$ if and only if there exists an $f \in X_0^{\#}$ such that

(i) $||f||_X = 1$

(ii) $f|_Y = 0$

iii) $|f(x) - f(y_0)| \geq d(x,y_0) - \varepsilon$.

Proof. Suppose $y_0 \in L_Y(x,\varepsilon)$. Define $f(z) = d(z,Y)$, $z \in X$. Then from the proof of Proposition 1 [3] it follows that this function satisfies (i), (ii) and

$$\begin{aligned} | f(x) - f(y_0)| &= d(x,Y) \\ &\geq d(x,y_0) - \varepsilon. \end{aligned}$$

Conversely, let us suppose that there exists $f \in X_0^{\#}$ satisfying (i), (ii) and (iii).

Then

$$\begin{aligned} d(x,y_0) &\le \mid f(x)-f(y_0)\mid + \varepsilon \\ &= \mid f(x)-f(y)\mid + \varepsilon \text{ for all } y\in Y \\ &\le \|f\|_X d(x,y) + \varepsilon \text{ for all } y\in Y \\ &= d(x,y) + \varepsilon \text{ for all } y\in Y. \end{aligned}$$

This implies that $d(x,y_0) \le d(x,Y)+\varepsilon$ i.e. $y_0 \in L_Y(x,\varepsilon)$.

Remarks. If Y is a subspace of a linear metric space (X,d) with a translation invariant metric d and $x_0 = 0$, the additive identity of X, then one can choose the function f in the preceeding discussion such that $f \in X^v$, where

$$X^v = \{f : X \longrightarrow \mathbb{R} : \sup_{x\in X\setminus\{0\}} \frac{\mid f(x)\mid}{d(x,0)} < \infty,\ f(0)=0,\ f \text{ subadditive}\},$$

is the cone of subadditive functions in X_0 (see [5] or [6]). If X is a normed linear space then $f \in X^*$ (see [7]).

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