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TOTALLY UMBILICAL SUBMANIFOLDS OF A COMPLEX SPACE FORM

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Recently Yamaguchi and others [3] have classified extrinsic spheres of a Kaehler manifold. Using their result, in this paper we classify the totally umbilical submanifolds of a complex space form. Our main result is the following:

Theorem: *Let M be an n-dimensional $(n > 2)$ complete, simply connected, totally umbilical submanifold of a $2m$-dimensional complex space form $\overline{M}(c)$. Then M is one of the following*

(i) *A complex space form $M(c)$*
(ii) *Totally real submanifold of constsnt curvature*
(iii) *Isometric to an ordinary sphere*
(iv) *Homothetic to a sasakian manifold.*

1. Preliminaries

Let $\overline{M}(c)$ be a $2m$-dimensional complex space form i.e. a Kaehler manifold of constant holomorphic sectional curvature c. The curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ]. \quad (1.1)$$

Let M be an n-dimensional submanifold of M. Then the Riemannian connection $\overline{\nabla}$ of $\overline{M}$ gives rise to a connection ∇ on M and a connection $\nabla^{\perp}$ in the normal bundle ν of M. The Gausś and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (1.2)$$
$$\overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N. \quad (1.3)$$

where X, Y are vector fields on $M, N \,\varepsilon\, \nu$ and h, A_N are second fundamental forms connected by

$$g(h(X,Y),N) = g(A_N X, Y). \quad (1.4)$$

The submanifold M is said to be totally umbilical if

$$h(X,Y) = g(X,Y)H \quad (1.5)$$

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where $H = \frac{1}{n}$ (trace h), is called the mean curvature vector. If $h = 0$, then M is said to be totally geodasic and if $H = 0$, then M is said to be minimal. For totally umbilical submanifold, they are equivalent. The equations of Gauss and Codazzi for totally umbilical submanifold in $\overline{M}(c)$ are

$$\begin{aligned} R(X,Y,Z,W) =& [\tfrac{c}{4} + g(H,H)][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \tfrac{c}{4}[g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) + 2g(X,JY)g(JZ,W)] \end{aligned} \qquad (1.6)$$

$$[\overline{R}(X,Y)Z]^{\perp} = g(Y,Z)\,\nabla_X^{\perp} H - g(X,Z)\,\nabla_Y^{\perp} H, \qquad (1.7)$$

where $[\ ]^{\perp}$ denotes the normal component of $\overline{R}(X,Y)Z$.

2. Proof of the theorem.

As $n > 2$, for every vector field X on M we can choose a vector field Y on M orthogonal to both X and JX. Thus (1.7) gives

$$[R(X,Y)Y]^{\perp} = g(Y,Y)\,\nabla_X^{\perp} H. \qquad (2.1)$$

On the other hand from (1.1) we have

$$R(X,Y)Y = \frac{c}{4} g(Y,Y)X. \qquad (2.2)$$

From (2.1) and (2.2) we get

$$\nabla_X^{\perp} H = 0, \text{ for every vector } X \text{ on } M. \qquad (2.3)$$

If $H = 0$, then M could be complex submanifold (as complex submanifolds of Kaehler manifold are minimal), we get part (i) of the theorem by (1.6).
If $H \neq 0$, then certainly M is not complex submanifold but it is an extrinsic sphere by (2.3). Then rest of our theorem follows from [3].

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