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AN INTEGRABILITY THEOREM FOR POWER SERIES

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1. Concerning integrability of a power series we prove the following theorem:

Theorem. *Let $\lambda(t)$ be a positive function defined on $(0,1]$ such that $\lambda(t)t^{-\delta}$ is non-increasing for some $\delta > 0$ and*

$$\sum_{n=k}^{\infty} \lambda(1/n)n^{-2} \leq M\lambda(1/k)k^{-1}. \tag{1.1}$$

Let $f(x) = \Sigma_0^\infty c_n x^n$, $0 \leq x < 1$. Suppose $\{\rho_n\}$ is a positive increasing sequence with $\Sigma_1^\infty \frac{1}{n\rho_n} < \infty$ such that

$$c_n > \frac{-K}{(\rho_n \lambda(1/n))^{1/p} n^{1-1/p}},\ 0 < p < \infty,\ K > 0 \tag{1.2}$$

for all sufficiently large n. Then $\lambda(1-x)|f(x)|^p \in L(0,1)$ iff

$$\sum_{n=1}^{\infty} \lambda(1/n)n^{-2}\left(\sum_{k=0}^{n} | c_k |\right)^p < \infty. \tag{1.3}$$

This generalizes a theorem of Leindler [4], where he assumes that $\lambda(t)$ is a non-increasing function. Also our theorem enables us to deduce a theorem of Jain [1] for all $\gamma < 1$ by choosing $\lambda(t) = t^{-\gamma}$ with $\delta > -\gamma$, whereas Leindler's theorem includes her theorem only for $0 \leq \gamma < 1$.

2. We need the following lemma for the proof of our theorem.

Lemma. *Let $\lambda(t)$ be a positive function such that $\lambda(t) \in L(0,1)$, $\lambda(t)t^{-\delta}$ is non-increasing for some $\delta > 0$ and $\lambda(\frac{1}{n+1}) = O(\lambda(1/n))$. Let $F(x) = \Sigma_0^\infty a_k x^k, 0 \leq x < 1, a_k \geq 0$ then for $0 < p \leq \infty$*

$$\lambda(1-x)F^p(x) \in L(0,1) \text{ iff} \tag{2.1}$$

$$\sum_{n=1}^{\infty} \lambda(1/n)n^{-2}S_n^p < \infty \text{ where } S_n = \sum_{k=0}^{n} a_k. \tag{2.2}$$

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This generalizes a previous theorem of Leindler [3].

Proof of the lemma. The proof is similar to that of Khan [2] or Leindler [3]. Since $(1-1/n)^n$ is increasing we have for $\frac{1}{n+1} \le y \le \frac{1}{n}, n \ge 2$

$$F(1-y) \ge \sum_0^n a_k(1-y)^k \ge \sum_{k=0}^n a_k(1-\frac{1}{n})^n \ge \frac{S_n}{4}.$$

Hence

$$\begin{aligned}\sum_{n=1}^\infty \lambda(\frac{1}{n})n^{-2}S_n^p &\le 2^{\delta+1}\sum_{n=1}^\infty \int_{1/n+1}^{1/n} \lambda(y)y^{-\delta}y^\delta S_n^p dy \\ &= O(1) + O(\sum_{n=2}^\infty \int_{1/n+1}^{1/n} \lambda(y)(F(1-y))^p dy) \\ &= O(\int_0^1 \lambda(1-x)F^p(x)dx) + O(1).\end{aligned}$$

Thus (2.1) $\Rightarrow$ (2.2).

To prove the converse we have

$$\int_0^1 \lambda(1-x)F^p(x)dx = \sum_{n=1}^\infty \int_{1/n+1}^{1/n} \lambda(y)(\sum_{k=0}^\infty a_k(1-y)^k)^p dy$$

$$\le \sum_{n=1}^\infty \int_{1/n+1}^{1/n} \lambda(y)\Big(\sum_{k=0}^\infty a_k(1-\frac{1}{n+1})^k\Big)^p dy$$

$$\le \sum_{n=1}^\infty \lambda(\frac{1}{n+1})(\frac{n+1}{n})^\delta\Big(\sum_{k=0}^\infty a_k(1-\frac{1}{n+1})^k\Big)^p \frac{1}{n(n+1)}$$

$$= O(\sum_{n=1}^\infty \lambda\Big(\frac{1}{n}\Big)n^{-2}\Big(\sum_{k=0}^\infty a_k(1-\frac{1}{n+1})^k\Big)^p\Big) \tag{2.3}$$

since $\lambda(\frac{1}{k+1}) = O(\lambda(1/k))$ which follows in view of (1.1) and the fact that

$$\sum_{n=k}^\infty \lambda(\frac{1}{n})n^{-2} \ge \lambda(\frac{1}{k+1})(k+1)^\delta \sum_{n=k+1}^\infty n^{-2-\delta} \ge \frac{1}{1+\delta}(1+k)^{-1}\lambda(\frac{1}{k+1}).$$

Suppose $p \ge 1$, then proceeding as in [4], we have

$$\int_0^1 \lambda(1-x)F^p(x)dx = O(1)\sum_{n=1}^\infty \lambda(\frac{1}{n})n^{-2}\sum_{i=1}^\infty 2^{-\frac{ip}{2}} S_{ni}^p$$

$$= O(1)\sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} \sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} S_{ni}^p = O(1)\sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} i^{2+\delta} \sum_{n=1}^{\infty} \lambda(\frac{1}{ni})(ni)^{-2} S_{ni}^p$$

$$= O(1)\sum_{i=1}^{\infty} i^{2+\delta} 2^{-\frac{ip}{2}} \sum_{m=1}^{\infty} \lambda(\frac{1}{m}) m^{-2} S_m^p = O(1)\sum_{m=1}^{\infty} \lambda(\frac{1}{m}) m^{-2} S_m^p,$$

since $\sum_{i=1}^{\infty} 2^{-\frac{ip}{2}} i^{2+\delta} < \infty$.

Now suppose $0 < p < 1$. Then

$$\sum_{k=0}^{\infty} a_k (1 - \frac{1}{n+1})^k \le S_n + \sum_{k=n}^{\infty} e^{-\frac{k}{n+1}} a_k$$

$$\le S_n + \sum_{j=1}^{\infty} \sum_{k=jn}^{(j+1)n-1} a_k e^{-\frac{k}{n+1}} \le S_n + \sum_{j=1}^{\infty} e^{-\frac{jn}{n+1}} \sum_{k=jn}^{(j+1)n-1} a_k$$

$$\le S_n + K\sum_{j=1}^{\infty} e^{-j/2} S_{(j+1)n}.$$

Hence from (2.3)

$$\int_0^1 \lambda(1-x) F^p(x) p\,dx$$

$$= O(\sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} S_n^p) + O(\sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} \sum_{j=1}^{\infty} e^{-\frac{pj}{2}} S_{(j+1)n}^p)$$

$$= O(\sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} S_n^p) + O(\sum_{j=1}^{\infty} e^{-\frac{pj}{2}} \sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} S_{(j+1)n}^p)$$

$$= O(\sum_{n=1}^{\infty} \lambda(\frac{1}{n}) n^{-2} S_n^p) + O(\sum_{i=2}^{\infty} e^{-\frac{p(i-1)}{2}} i^{2+\delta} \sum_{m=1}^{\infty} \lambda(\frac{1}{m}) m^{-2} S_m^p)$$

$$= O(\sum_{m=1}^{\infty} \lambda(\frac{1}{m}) m^{-2} S_m^p).$$

Thus (2.2) $\Rightarrow$ (2.1).

3. Proof of the theorem. Let $A(x) = \Sigma_0^\infty b_n x^n$, $0 \le x < 1$ with

$$b_n = \frac{K}{(\rho_n \lambda(1/n))^{1/p} n^{1-\frac{1}{p}}}, \quad n \ge 1.$$

Using the inequality [4]

$$\sum_{n=1}^{\infty}\lambda_n(\sum_{k=1}^{n}b_k)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}(\sum_{k=n}^{\infty}\lambda_k)^p b_n^p$$

which holds for any $\lambda_n > 0$, $b_n \ge 0$, $p \ge 1$, we have with $\lambda_n = \lambda(1/n)n^{-2}$ and in view of (1.1)

$$\sum_{n=1}^{\infty}\lambda(1/n)n^{-2}(\sum_{k=1}^{n}b_k)^p \le O(1)\sum_{1}^{\infty}(\lambda(\frac{1}{n})n^{-2})^{1-p}(\lambda(\frac{1}{n})n^{-1})^p b_n^p$$

$$=O(\sum_{1}^{\infty}\lambda(\frac{1}{n})n^{-2+p}b_n^p) = O(\sum_{1}^{\infty}\lambda(\frac{1}{n})n^{-2+p}\frac{1}{\rho_n\lambda(\frac{1}{n})n^{p-1}}) = O(\sum_{n=1}^{\infty}\frac{1}{n\rho_n}) = O(1).$$

If $0 < p < 1$, then

$$\sum_{n=2}^{\infty}\lambda(\frac{1}{n})n^{-2}(\sum_{k=1}^{n}b_k)^p \le \sum_{m=1}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda(\frac{1}{n})n^{-2}(\sum_{k=3}^{2^{m+1}}b_k)^p + O(1)$$

$$\le 2^{\delta}\sum_{m=1}^{\infty}\lambda(\frac{1}{2^{m+1}})2^{-m}(\sum_{k=3}^{2^{m+1}}b_k)^p + O(1).$$

Now

$$\sum_{k=3}^{2^{m+1}}b_k = K\sum_{k=1}^{m}\sum_{n=2^k+1}^{2^{k+1}}\frac{n^{\frac{1}{p}-1+\frac{\delta}{p}}}{(\rho_n\lambda(\frac{1}{n})n^{\delta})^{1/p}}$$

$$\le K\sum_{k=1}^{m}\frac{2^{(k+1)(\frac{1}{p}-1+\frac{\delta}{p})}2^k}{(\rho_{2^k}\lambda(\frac{1}{2^k})2^{k\delta})^{1/p}} \le K\sum_{k=1}^{m}\frac{2^{k/p}}{(\rho_{2^k}\lambda(\frac{1}{2^k}))^{1/p}}.$$

Hence

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\lambda(\frac{1}{n})n^{-2}(\sum_{k=1}^{n}b_k)^p \\
&\leq K\sum_{m=1}^{\infty}\lambda(\frac{1}{2^{m+1}})2^{-m}\sum_{k=1}^{m}\frac{2^k}{\rho_{2^k}\lambda(\frac{1}{2^k})}+O(1) \\
&= K\sum_{k=1}^{m}\frac{2^k}{\rho_{2^k}\lambda(\frac{1}{2^k})}\sum_{m=k}^{\infty}2^{-m}\lambda(\frac{1}{2^{m+1}})+O(1) \\
&\leq K\sum_{k=1}^{\infty}\frac{2^k}{\rho_{2^k}\lambda(1/2^k)}\lambda(1/2^k)2^{-k}+O(1) \\
&= K\sum_{k=1}^{\infty}\frac{1}{\rho_{2^k}}+O(1) \\
&\leq K\sum_{n=1}^{\infty}\frac{1}{n\rho_n}+O(1) \\
&= O(1),
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{n=k}^{\infty}2^{-n}\lambda(\frac{1}{2^n}) \ &\leq \ 2^{2+\delta}\sum_{n=k}^{\infty}\sum_{m=2^n+1}^{2^{n+1}} m^{-2}\lambda(\frac{1}{m}) \\
&= \ 2^{2+\delta}\sum_{m=2^k+1}^{\infty} m^{-2}\lambda(\frac{1}{m}) \ \leq \ K\lambda(\frac{1}{2^k})2^{-k}
\end{aligned}
$$

by (1.1).

Thus for $0 < p < \infty$ the condition (2.2) of the lemma is satisfied. Hence

$$\lambda(1-x)(A(x))^p \in L(0,1). \tag{3.4}$$

Now proceeding exactly as in [4] we complete the proof of the theorem.

References

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