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ON THE TOTAL CURVATURE OF SURFACES IMMERSED IN EUCLIDEAN SPACES OF DIMENSION HIGHER THAN FOUR

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1. Introduction

The total absolute curvature $\tau(M)$ of a smoothly immersed n-submanifold M of a Euclidean $(n+N)$-space E^{n+N} was first studied by S.S.Chern and R.K.Lashof ([5]) and then by N.H.Kuiper ([6]) through *the Lipschitz-Killing curvature* $K(x,e)$ defined by the dual map $\tilde{\nu}^*$ of the Gauss map $\tilde{\nu}: B_\nu \longrightarrow S_0^{n+N-1}$ of the unit normal bundle B_ν over M into the unit $(n+N-1)$-sphere S_0^{n+N-1} of the center at the origin in E^{n+N} at each point (x,e) of the bundle B_ν such that

$$\tilde{\nu}^*\, d\sigma_{n+N-1} \;=\; K(x,e)\, d\sigma_{N-1}\Lambda dv, \tag{1.1}$$

where dv and $d\sigma_m$ are the volume elements of M and an m-sphere S^m, respectively.

The geometric meaning of the Lipschitz-Killing curvature $K(x,e)$ is described in detail by Y.T.Shin ([8]) as a generalization of the Gauss-Kronecker curvature of a hypersurface M in E^{n+1} or the Gauss curvature of M in E^3. *The total absolute curvature* $K^*(x) = \int_{S^{N-1}} \mid K(x,e) \mid d\sigma_{N-1}$ *at each point* x of M is defined as the integral of the absolute value of the Lipschitz-Killing curvature $K(x,e)$ over each fiber of the unit normal bundle B_ν over M, and *the total absolute curvature* $\tau(M) = \int_M K^*(x)dv$ of M as the integral of $K^*(x)$ over M if it exists.

One of results Chern-Lashof and Kuiper proved in their first papers, applying the Morse inequality ([7]), is

$$\tau(M) \;\geq\; C_{n+N-1}\beta(M), \tag{1.2}$$

where C_{n+N-1} is the volume of $(n+N-1)$-sphere S^{n+N-1} and $\beta(M)$ is the sum of the betti numbers of M. The right-hand side of (1.2) depends on the coefficient field. If the equality sign holds in (1.2) with the real field as coefficient field, then M has zero torsion. And we know the Gauss-Bonnet theorem for a compact surface M in E^m

$$\int_M G\, dv \;=\; 2\pi x(M), \tag{1.3}$$

where G is the Gauss curvature and $x(M)$ is the Euler characteristic of M.

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Some of the results we have proved in this paper are the following.
If M is a compact flat surface in E^m with $\lambda_2 \leq 0$, then we have

$$\int_M \alpha^2\, dv \;\geq\; \frac{\pi^2}{2}\beta(M). \tag{1.4}$$

Let M be a compact surface in E^5. Then we have

$$\tau(M) \;\leq\; \frac{8}{3}\pi^2\chi(M) \;-\; 8\pi\int_M \lambda_3(x)\, dv. \tag{1.5}$$

Concering the total absolute curvature $\tau(M)$ and the total mean curvature $\int_M \alpha^2 dv$ of a compact surface in E^5 with $\lambda_1 \geq 0$, we have proved the inequality

$$\int_M \alpha^2\, dv \;\geq\; \frac{1}{24\pi}(3\tau(M) \;+\; 8\pi^2\chi(M)). \tag{1.6}$$

Let M be a flat surface in E^5 with $\lambda_2 \geq 0$. Then we have

$$\int_M \alpha^2\, dv \;\geq\; \frac{3}{16\pi}\,\tau(M). \tag{1.7}$$

2. Preliminaries

Let M be a surface in a Euclidean m-space E^m, $m \geq 5$. We choose a local field of orthonormal frames $e_1, e_2, \ldots, e_m$ in E^m such that, restricted to M, e_1, e_2 are tangent to M and $e_3, \ldots, e_m$ are normal to M. Let $\omega^1, \omega^2, \cdots, \omega^m$ be the field of dual frame. Then the structure equations of E^m are given by

$$\begin{aligned} d\omega^A \;&=\; -\sum_B \omega^A{}_B \wedge \omega^B, \\ \omega^A{}_B + \omega^B{}_A \;&=\; 0, \\ d\omega^A{}_B \;&=\; -\sum_C \omega^A{}_C \wedge \omega^C{}_B, \qquad A,\, B,\, C \;=\; 1,\, 2,\, \cdots,\, m. \end{aligned}$$

We restrict these forms to M. Then $\omega^3 \;=\; \cdots \;=\; \omega^m \;=\; 0$.
Since

$$0 \;=\; d\omega^r \;=\; -\sum_i \omega^r{}_i \wedge \omega^i, \qquad i,\, j,\, k \;=\; 1,\, 2, \qquad r,\, s,\, t \;=\; 3,\, 4,\, \cdots,\, m,$$

by Cartan's lemma, we may write

$$\omega^r{}_i \;=\; \sum_j h^r_{ij}\, \omega^j, \qquad h^r_{ij} \;=\; h^r_{ji}.$$

We call $h = \sum_{r,i,j} h^r_{ij}\,\omega^i\omega^j e_r$ *the second fundamental form* of M. The mean curvature vector H is given by

$$H = \frac{1}{2}\sum_r (h^r_{11} + h^r_{22})\,e_r.$$

They are generalized cases of the surfaces in E^3.
If $H = 0$, then M is called *a minimal surface.* In [10], it is proved that there does not exist a closed minimal submanifold in a Euclidean space.

The Gauss curvature G and *the mean curvature* α are defined respectively by

$$G = \sum_{r=3}^{m}(h^r_{11}h^r_{22} - h^r_{12}h^r_{12}),$$

$$\alpha = \frac{1}{2}\Big(\sum_{r=3}^{m}(h^r_{11} + h^r_{22})^2\Big)^{\frac{1}{2}}. \tag{2.1}$$

For a normal vector $e = \sum_{r=3}^{m} a_r e_r$ at x in M, the second fundamental tensor $A(x,e)$ at (x,e) is given by

$$A(x,e) = \sum_{r=3}^{m} a_r h^r_{ij}.$$

The Lipschitz-Killing curvature $K(x,e)$ is defined by

$$\begin{aligned} K(x,e) &= \det\,(A(x,e)) \\ &= \Big(\sum_{r=3}^{m} a_r h^r_{11}\Big)\Big(\sum_{s=3}^{m} a_s h^s_{22}\Big) - \Big(\sum_{t=3}^{m} a_t h^t_{12}\Big)^2. \end{aligned}$$

For each x in M, we denote by $T_x^{\perp}$ the normal space at x. We define a linear mapping γ from $T_x^{\perp}$ into the space of all symmetric matrices of order 2 by

$$\gamma\Big(\sum_{r=3}^{m} a_r e_r\Big) = \sum_{r=3}^{m} a_r A(x,e_r).$$

Then, since $\dim T_x^{\perp} = \dim \ker \gamma + \dim \operatorname{Im} \gamma$,

$$\dim \ker \gamma \geq m - 5.$$

We choose $e_3, e_4, \ldots, e_m$ at x in such a way that $e_6, \ldots, e_m \in \ker \gamma$. Then for any unit normal vector $e = \sum_r \cos\theta_r e_r$ at x, the Lipschitz-Killing curvature $K(x,e)$ at (x,e) is given by

$$\begin{aligned} K(x,e) &= \det\,(A(x,e)) \\ &= \det\Big(\sum_{r=3}^{m} h^r_{ij}\cos\theta_r\Big) \\ &= \Big(\sum_{r=3}^{5} h^r_{11}\cos\theta_r\Big)\Big(\sum_{s=3}^{5} h^s_{22}\cos\theta_s\Big) - \Big(\sum_{t=3}^{5} h^t_{12}\cos\theta_t\Big)^2. \end{aligned} \tag{2.2}$$

The right hand side of (2.3) is a quadratic form on $\cos\theta_r$. Hence, by choosing a suitable unit orthogonal normal vectors e_3, e_4, e_5 at x, we may write

$$K(x,e) = \lambda_1(x)\cos^2\theta_3 + \lambda_2(x)\cos^2\theta_4 + \lambda_3(x)\cos^2\theta_5, \quad \lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x). \tag{2.3}$$

From now on, we choose such a frame unless otherwise stated.
From (2.1), we obtain

$$G = \lambda_1 + \lambda_2 + \lambda_3 \text{ everywhere on } M. \tag{2.4}$$

Let C_m be the volume of the unit m-sphere S^m. Then we know that

$$C_m = \frac{2\,\pi^{\frac{1}{2}(m+1)}}{\Gamma(\frac{1}{2}(m+1))}, \tag{2.5}$$

where Γ is the Gamma function.

By spherical integration ([9]), we obtain the following equations (2.6) and (2.7).

$$\int_{S^m} |\cos^2\theta_r - \cos^2\theta_s|\, d\sigma = \frac{2C_{m+2}}{\pi^2}, \qquad r \neq s, \tag{2.6}$$

where $d\sigma$ is the volume element of the unit m-sphere S^m.

$$\int_{S^m} \cos^2\theta_r\, d\sigma = \frac{C_{m+2}}{2\pi}. \tag{2.7}$$

3. Main Results

We begin with the following lemma which is crucial for our argument.

Lemma 3.1. *Let M be a surface in E^m, $m \geq 5$. Then $\lambda_3 \leq 0$ everywhere on M.*

Proof. Let e be a unit normal vector at $x \in M$ which is perpendicular to the mean curvature vector H. Then

$$H \cdot e = \frac{1}{2}\sum_r (h_{11}^r + h_{22}^r)\cos\theta_r = 0,$$

where $e = \sum_r \cos\theta_r e_r$. Hence

$$\Big(\sum_{r=3}^{5} h_{11}^r\cos\theta_r + \sum_{s=3}^{5} h_{22}^s\cos\theta_s\Big)^2 = \Big(\sum_{r=3}^{5} h_{11}^r\cos\theta_r\Big) + \Big(\sum_{s=3}^{5} h_{22}^s\cos\theta_s\Big)^2$$
$$+ 2\Big(\sum_{r=3}^{5} h_{11}^r\cos\theta_r\Big)\Big(\sum_{s=3}^{5} h_{22}^s\cos\theta_s\Big) = 0.$$

Therefore the Lipschitz-Killing curvature $K(x,e)$ at (x,e) is given by

$$K(x,e) = (\sum_{r=3}^{5} h^r_{11} \cos\theta_r)(\sum_{s=3}^{5} h^s_{22} \cos\theta_s) - (\sum_{t=3}^{5} h^t_{12} \cos\theta_t)^2 \leq 0.$$

Thus, from view points of (2.3), we complete the proof.

Theorem 3.2. *Let M be a compact surface in E^m with $\lambda_3 = 0$. Then M is homeomorphic to a 2-sphere.*

Proof. Let S_x be the unit hypersphere of $T_x^{\perp}$ and let $d\sigma$ be the volume element of S_x. From (2.3) and (2.7), we have

$$\begin{aligned} K^*(x) &= \int_{S_x} \mid \lambda_1(x)\cos^2\theta_3 + \lambda_2(x)\cos^2\theta_4 \mid d\sigma \\ &= (\lambda_1(x) + \lambda_2(x))\frac{C_{m-1}}{2\pi} = \frac{C_{m-1}}{2\pi}G(x), \end{aligned}$$

where C_{m-1} denotes the volume of the unit $(m-1)$-sphere. Hence the total absolute curvature $\tau(M)$ of M is given by

$$\begin{aligned} \tau(M) &= \frac{C_{m-1}}{2\pi}\int_M G(x)\,dv \\ &= C_{m-1}\chi(M) \geq C_{m-1}\beta(M) \end{aligned}$$

by (1.2) and (1.3), where $\chi(M)$ denotes the Euler characteristic of M. Therefore $\chi(M) \geq \beta(M)$. Since $\chi(M) \leq \beta(M)$, $\chi(M) = \beta(M)$. Thus, by the arguments in the inequality (1.2), M has zero torsion and $\chi(M) = 2$. Hence M is homeomorphic to a 2-sphere.

Lemma 3.3. *If M is a compact flat surface in E^m with $\lambda_2 \leq 0$, then we have*

$$\int_M \lambda_1(x)\,dv \geq \frac{\pi^2}{2}\beta(M).$$

Proof. Since $\lambda_1 = -\lambda_2 - \lambda_3$,

$$K(x,e) = \lambda_2(x)(\cos^2\theta_4 - \cos^2\theta_3) + \lambda_3(x)(\cos^2\theta_5 - \cos^2\theta_3).$$

Hence

$$\begin{aligned} K^*(x) &\leq -\lambda_2(x)\int_{S_x} \mid \cos^2\theta_4 - \cos^2\theta_3 \mid d\sigma - \lambda_3(x)\int_{S_x} \mid \cos^2\theta_5 - \cos^2\theta_3 \mid d\sigma \\ &= -(\lambda_2(x) + \lambda_3(x))\frac{2C_{m-1}}{\pi^2} = 2\lambda_1(x)\frac{C_{m-1}}{\pi^2}, \end{aligned}$$

by (2.6). Therefore, by (1.2),

$$C_{m-1}\beta(M) \le \tau(M) \le \frac{2C_{m-1}}{\pi^2}\int_M \lambda_1(x)dv.$$

Thus

$$\int_M \lambda_1(x)\, dv \ge \frac{\pi^2}{2}\beta(M).$$

Lemma 3.4. *Let M be a surface in E^m. Then we have $\alpha^2 \ge \lambda_1$.*

Proof.

$$\begin{aligned}4\alpha^2 &= (h_{11}^3)^2 + (h_{22}^3)^2 + (h_{11}^4)^2 + (h_{22}^4)^2 + (h_{11}^5)^2 + (h_{22}^5)^2 \\ &\qquad + 2(h_{12}^3)^2 + 2(h_{12}^4)^2 + 2(h_{12}^5)^2 + 2G \\ &\ge 2h_{11}^3h_{22}^3 - 2h_{11}^4h_{22}^4 - 2h_{11}^5h_{22}^5 + 2(h_{12}^3)^2 + 2(h_{12}^4)^2 + 2(h_{12}^5)^2 + 2G \\ &\ge 2\lambda_1 - 2\lambda_2 - 2\lambda_3 + 2G = 4\lambda_1.\end{aligned}$$

Theorem 3.5. *Let M be a compact flat surface in E^m with $\lambda_2 \le 0$. Then we have*

$$\int_M \alpha^2\, dv \ge \frac{\pi^2}{2}\beta(M).$$

Proof. By Lemma 3.3 and Lemma 3.4,

$$\int_M \alpha^2\, dv \ge \int_M \lambda_1(x)\, dv \ge \frac{\pi^2}{2}\beta(M).$$

Lemma 3.6. *Let M be a surface in E^5. Then we have*

$$K^*(x) \le \frac{4\pi}{3}G(x) - 8\pi\,\lambda_3(x).$$

The equality sign holds when and only when $\lambda_1(x) = \lambda_2(x) = \lambda_3(x)$ or $\lambda_3(x) = 0$.

Proof. Since $\Sigma_{r=3}^5 \cos^2\theta_r = 1$, by (2.3),

$$\begin{aligned}K(x,e) &= \lambda_1(x)\cos^2\theta_3 + \lambda_2(x)\cos^2\theta_4 + \lambda_3(x)(1-\cos^2\theta_3 - \cos^2\theta_4) \\ &= (\lambda_1(x)-\lambda_3(x))\cos^2\theta_3 + (\lambda_2(x)-\lambda_3(x))\cos^2\theta_4 + \lambda_3(x).\end{aligned}$$

By (2.5) and (2.7),

$$\int_{S_x} \cos^2\theta_r\, d\sigma = \frac{C_4}{2\pi} = \frac{4\pi}{3}.$$

Therefore

$$\begin{aligned} K^*(x) &\le (\lambda_1(x) - \lambda_3(x))\frac{4\pi}{3} + (\lambda_2(x) - \lambda_3(x))\frac{4\pi}{3} - \lambda_3(x)\,4\pi \\ &= \frac{4\pi}{3}G(x) - 8\pi\,\lambda_3(x). \end{aligned}$$

If $K^*(x) = \dfrac{4\pi}{3}G(x) - 8\pi\lambda_3(x)$, then

$$|\,K(x,e)\,| = (\lambda_1(x) - \lambda_3(x))\cos^2\theta_3 + (\lambda_2(x) - \lambda_3(x))\cos^2\theta_4 - \lambda_3(x)$$

for all θ_3, θ_4. Hence

$$\lambda_1(x) = \lambda_2(x) = \lambda_3(x) \text{ or } \lambda_3(x) = 0.$$

The converse of this is trivial.

From Lemma 3.6 and (1.3), we obtain the following.

Theorem 3.7. *Let M be a compact surface in E^5. Then we have*

$$\tau(M) \le \frac{8}{3}\pi^2\,\chi(M) - 8\pi\int_M \lambda_3(x)\,dv,$$

where $\chi(M)$ is the Euler characteristic of M.

Lemma 3.8. *Let M be a surface in E^5 with $\lambda_1 \ge 0$. Then we have*

$$K^*(x) \le 8\pi\lambda_1(x) - \frac{4\pi}{3}G(x).$$

The equality sign holds when and only when $\lambda_1(x) = 0$.

Proof. Since $\Sigma_{r=3}^5 \cos^2\theta_r = 1$,

$$\begin{aligned} K(x,e) &= \lambda_1(x)(1 - \cos^2\theta_4 - \cos^2\theta_5) + \lambda_2(x)\cos^2\theta_4 + \lambda_3(x)\cos^2\theta_5 \\ &= \lambda_1(x) + (\lambda_2(x) - \lambda_1(x))\cos^2\theta_4 + (\lambda_3(x) - \lambda_1(x))\cos^2\theta_5. \end{aligned}$$

Hence

$$\begin{aligned} K^*(x) &\le \lambda_1(x)C_2 - (\lambda_1(x) + \lambda_2(x) + \lambda_3(x))\frac{C_4}{2\pi} + 3\lambda_1(x)\frac{C_4}{2\pi} \\ &= 8\pi\lambda_1(x) - \frac{4\pi}{3}G(x). \end{aligned}$$

If the equality holds, then we must have

$$|\,K(x,e)\,| = \lambda_1(x) + (\lambda_1(x) - \lambda_2(x))\cos^2\theta_4 + (\lambda_1(x) - \lambda_3(x))\cos^2\theta_5$$

for all θ_4 and θ_5. Therefore

$$\lambda_1(x) = 0 \text{ or } \lambda_1(x) = \lambda_2(x) = \lambda_3(x).$$

But the second condition also implies $\lambda_1(x) = 0$, since $\lambda_1(x) \geq 0$ and $\lambda_3(x) \leq 0$. The converse of this is trivial.

Theorem 3.9. *Let M be a compact surface in E^5 with $\lambda_1 \geq 0$. Then we have*

$$\int_M \alpha^2\, dv \geq \frac{1}{24\pi}(3\tau(M) + 8\pi^2\chi(M)).$$

Proof. From Lemma 3.4 and Lemma 3.8,

$$\begin{aligned}\int_M \alpha^2\, dv &\geq \int_M \lambda_1(x)\, dv\\ &\geq \frac{1}{8\pi}\int_M (K^*(x) + \frac{4\pi}{3}G(x))\, dv\\ &= \frac{1}{8\pi}(\tau(M) + \frac{8\pi^2}{3}\chi(M)).\end{aligned}$$

Lemma 3.10. *Let M be a surface in E^m. Then we have*

$$\alpha^2 \geq \lambda_1 + \lambda_2.$$

Proof.

$$\begin{aligned}4\alpha^2 &= (h_{11}^3)^2 + (h_{22}^3)^2 + (h_{11}^4)^2 + (h_{22}^4)^2 + (h_{11}^5)^2 + (h_{22}^5)^2\\ &\quad + 2(h_{12}^3)^2 + 2(h_{12}^4)^2 + 2(h_{12}^5)^2 + 2G\\ &\geq 2h_{11}^3h_{22}^3 + 2h_{11}^4h_{22}^4 - 2h_{11}^5h_{22}^5 + 2(h_{12}^3)^2 + 2(h_{12}^4)^2 + 2(h_{12}^5)^2 + 2G\\ &\geq 2\lambda_1 + 2\lambda_2 - 2\lambda_3 + 2G\\ &= 4(\lambda_1 + \lambda_2).\end{aligned}$$

Lemma 3.11. *Let M be a surface in E^5 with $\lambda_2 \geq 0$. Then we have*

$$K^*(x) \leq 4\pi\lambda_2(x) + \frac{4\pi}{3}(\lambda_1(x) - \lambda_3(x)).$$

The equality sign holds when and only when $\lambda_1(x) = \lambda_2(x) = 0$ or $\lambda_2(x) = \lambda_3(x) = 0$.

Proof.
$$\begin{aligned}K(x,e) &= \lambda_1(x)\cos^2\theta_3 + \lambda_2(x)(1 - \cos^2\theta_3 - \cos^2\theta_5) + \lambda_3(x)\cos^2\theta_5\\ &= \lambda_2(x) + (\lambda_1(x) - \lambda_2(x))\cos^2\theta_3 + (\lambda_3(x) - \lambda_2(x))\cos^2\theta_5.\end{aligned}$$

Hence

$$\begin{aligned}K^*(x) &\leq \lambda_2(x)C_2 + (\lambda_1(x) - \lambda_2(x))\frac{4\pi}{3} + (\lambda_2(x) - \lambda_3(x))\frac{4\pi}{3}\\ &= 4\pi\lambda_2(x) + \frac{4\pi}{3}(\lambda_1(x) - \lambda_3(x)).\end{aligned}$$

If $K^*(x) = 4\pi\lambda_2(x) + \frac{4\pi}{3}(\lambda_1(x) - \lambda_3(x))$, then

$$| K(x,e) | = \lambda_2(x) + (\lambda_1(x) - \lambda_2(x))\cos^2\theta_3 + (\lambda_2(x) - \lambda_3(x))\cos^2\theta_5$$

for all θ_3, θ_5. Therefore

$$\lambda_1(x) = \lambda_2(x) = 0 \text{ or } \lambda_2(x) = \lambda_3(x).$$

But the second condition implies $\lambda_2(x) = \lambda_3(x) = 0$, since $\lambda_2(x) \geq 0$ and $\lambda_3(x) \leq 0$. The converse of this trivial.

Theorem 3.12. *Let M be a flat surface in E^5 with $\lambda_2 \geq 0$. Then we have $\int_M \alpha^2\, dv \geq \frac{3}{16\pi}\tau(M)$.*

Proof. Since $\lambda_3 = -\lambda_1 - \lambda_2$,

$$\begin{aligned} K^*(x) &\leq 4\pi\lambda_2(x) + \frac{4\pi}{3}(2\lambda_1(x) + \lambda_2(x)) \\ &= \frac{8\pi}{3}(\lambda_1(x) + 2\lambda_2(x)) \\ &= \frac{16\pi}{3}(\lambda_1(x) + \lambda_2(x)) - \frac{8\pi}{3}\lambda_1(x), \end{aligned}$$

by Lemma 3.11. From Lemma 3.10,

$$\begin{aligned} \int_M \alpha^2\, dv &\geq \int_M (\lambda_1(x) + \lambda_2(x))\, dv \\ &\geq \frac{3}{16\pi}\int_M (K^*(x) + \frac{8\pi}{3}\lambda_1(x))\, dv \\ &\geq \frac{3}{16\pi}\tau(M), \text{ since } \lambda_1(x) \geq 0. \end{aligned}$$

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