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UNIFIED CONTACT STRUCTURE

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Abstract. A necessary and sufficient condition that a C^∞ manifold to admits anunified contact structure is established. A condition is obtained so that it becomes P-Sasakian. A linear connection in it is defined so its distributions are parallel.

1. Introduction

If, on a manifold M, there exists a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \ldots, \xi_r$ and r 1-forms $\eta^1, \cdots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta\ ,\ \alpha, \beta \in (r) = \{1, \cdots, r\} \tag{1.1}$$

$$\phi^2 = a^2\{I - \eta^\alpha \otimes \xi_\alpha\} \tag{1.2}$$

where $a^\alpha b_\alpha$ denote $\sum a^\alpha b_\alpha$ where 'a' is a complex constant and I denotes identity operator

$$\eta^\alpha \circ \phi = 0, \qquad \alpha \in (r) \tag{1.3}$$

$$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r). \tag{1.4}$$

Then $\sum = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is called an unfield contact structure on M and M is an unified contact manifold.

If the structure ϕ is compatible with the Riemannian metric g, i.e.

$$g(\phi X, \phi Y) = \{g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\}. \tag{1.5}$$

For any vectors X and Y. Then $a^2 = \pm 1$. In case of $a^2 = -1$, the structure ϕ is almost contact and skew-symmetric. In case of $a^2 = 1$, the structure ϕ is almost paracontact and symmetric. If ϕ is parallel with respect to the Riemannian connection, then (g, ϕ) is Sasakian or para-contact structure respectively.

If one put the condition

$$g(\phi X, \phi Y) = \lambda\{g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)\} \tag{1.6}$$

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in place of the compatibility (1.5), then λ is positive because the metric g is positive definite, and $\lambda^2 = a^4$ or $\lambda = |a^2|$ by means of (1.6) and the structure condition (1.2).

An unified contact manifold M always admits the following complementary distributions.

$$D^+ = \{X;\ \phi(X) = aX\}, \tag{1.7}$$
$$D^- = \{X;\ \phi(X) = -aX\}, \tag{1.8}$$
$$D^0 = \{X;\ \phi(X) = 0\}, \tag{1.9}$$

we define operators

$$\phi_1 = \frac{1}{2}\{I + \frac{1}{a}\phi - \eta^\alpha \otimes \xi_\alpha\} = \frac{1}{2}\{\frac{1}{a^2}\phi^2 + \frac{1}{a}\phi\}, \tag{1.10}$$
$$\phi_2 = \frac{1}{2}\{I - \frac{1}{a}\phi - \eta^\alpha \otimes \xi_\alpha\} = \frac{1}{2}\{\frac{1}{a^2}\phi^2 - \frac{1}{a}\phi\}, \tag{1.11}$$
$$\phi_3 = \{I - \frac{1}{a^2}\phi^2\}, \tag{1.12}$$

They have following properties

$$\phi_1 + \phi_2 + \phi_3 = I, \tag{1.13}$$

$$\phi_1^2 = \phi_1, \qquad \phi_2^2 = \phi_2, \qquad \phi_3^2 = \phi_3, \tag{1.14}$$

$$\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 = \phi_1 \circ \phi_3 = \phi_3 \circ \phi_1 = \phi_2 \circ \phi_3 = \phi_3 \circ \phi_2 = 0 \tag{1.15}$$

and moreover,

$$\phi \circ \phi_1 = \phi_1 \circ \phi = a\phi_1, \tag{1.16}$$
$$\phi \circ \phi_2 = \phi_2 \circ \phi = -a\phi_2, \tag{1.17}$$
$$\phi \circ \phi_3 = \phi_3 \circ \phi = 0. \tag{1.18}$$

By (1.16), (1.17), (1.18) the distributions D^+, D^- and D^0 may be expressed as follows

$$\begin{aligned} D^+ &= \{X;\ \phi_1 X = X\} \\ D^- &= \{X;\ \phi_2 X = X\} \\ D^0 &= \{X;\ \phi_3 X = 0\} \end{aligned} \tag{1.19}$$

Thus we have Lemma,

Lemma 1 [3]. *The distributions D^+, D^-, D^0 are generated by the projection operators ϕ_1, ϕ_2, ϕ_3 respectively.*

Theorem 1. *The necessary and sufficient condition for M admits an unified contact structure is that there exists three complementary distributions D_1, D_2, D_3 of dimension p, q, r respectively, with $p + q + r = n = \dim M$.*

Proof. The necessary condition immediately follows from Lemma 1. Now suppose that there are three complementary distributions D_1, D_2, D_3 of dimension p, q, r respectively, with $p + q + r = \dim M = n$.

For any $X \in M$, we have $T_xM = D_1x + D_2x + D_3x$. Let

$$\{e_1, \ldots, e_p, e_{p+1} = \bar{e}_1, \ldots, e_{p+q} = \bar{e}_q, e_{p+q+1} = \xi_1, \ldots, e_n = \xi_r\}$$

be a basis of T_xM, where$\{e_i; i \in (P)\}$ is a basis for D_1x, $\{\bar{e}_t; t \in (q)\}$ is a basis for D_2x, and $\{\xi_1, \ldots, \xi_r\}$ is a basis for D_3x.

Now let $\{e^1, \cdots, e^p, e^{p+1} = \bar{e}^1, \cdots, e^{p+q} = \bar{e}^q, e^{p+q+1} = \eta^1, \cdots, e^n = \eta^r\}$ be a basis of cotangent space T_x^*M such that

$$e^i(e_j) = \delta^i_j, \qquad i, j \in (n) \tag{1.20}$$

which gives,

$$e^k \otimes e_k + \bar{e}^t \otimes \bar{e}_t + \eta^\alpha \otimes \xi_\alpha = I, \tag{1.21}$$

where

$$k \in (p), \qquad t \in (q), \qquad \alpha \in (r).$$

Let us put

$$\phi = a\{e^k \otimes e_k + \varepsilon \bar{e}^t \otimes \bar{e}_t\}, \qquad \varepsilon = \pm 1,\ k \in (p),\ t \in (q)$$

Then by virtue of (1.20) and (1.21), we get

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta$$

and

$$\phi^2 = a^2\{e^k \otimes e_k + \bar{e}^t \otimes \bar{e}_t\},$$

or

$$\phi^2 = a^2\{I - \eta^\alpha \otimes \xi_\alpha\}$$

Thus M admits an unified contact structure

$$(\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$$

Hence the proof.

2. On unified contact manifold M, with structure $\sum = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$

We define two operators

$$P = I \otimes I - \phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2 - \phi_3 \otimes \phi_3 \tag{2.1}$$

and

$$Q = \phi_1 \otimes \phi_1 + \phi_2 \otimes \phi_2 + \phi_3 \otimes \phi_3 \tag{2.2}$$

with properties

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0. \tag{2.3}$$

Lemma 2. [2]. *If A is a projection operators, i.e. $A^2 = A$ and $B = I - A$, then all the solutions of the equation $Ax = y$ are of the form $x = y + Bw$, where w is orbitrary.*

Definition. A distribution D on a manifold M is said to be parallel with respect to a given connection Γ. If, for every vector fields Y and X which belongs to the distribution D, the vector field $\nabla_Y X$ belongs to the distribution D, where ∇ is covariant derivative with respect to the connection Γ.

Proposition 1. [3]. *The distribution D^+ given by (1.7) is parallel with respect to a connection Γ, iff*

$$\nabla\phi_1 \circ \phi_1 = 0 \tag{2.4}$$

Proposition 2. [3]. *The distribution D^- is parallel with respect to a connection Γ iff*

$$\nabla\phi_2 \circ \phi_2 = 0 \tag{2.5}$$

Proposition 3. [3]. *The distribution D^0 is parallel with respect to a connection Γ iff*

$$\nabla\phi_3 \circ \phi_3 = 0 \tag{2.7}$$

Now if we assume that ∇ is an $\sum$- connection, then from (1.10), (1.11), (1.12) we get $\nabla\phi_1 = 0$, $\nabla\phi_2 = 0$, $\nabla\phi_3 = 0$.

Theorem 2. *If Γ is an $\sum$- connection on an unified contact manifold M, then the distribution D^+, D^-, D^0 given by (1.7), (1.8) and (1.9) are parallel with respect to this connection.*

Proof. By virtue of proposition 1, 2, and 3 we obtain the required proof.
Now we shall find all connections of the form

$$\overline{\nabla}_X = \nabla_X + A_x \tag{2.8}$$

with respect to which the distributions D^+, D^-, D^0 given by (1.7), (1.8) and (1.9) are parallel, where ∇ is the co-derivative with respect to any orbitrary connection Γ on an unified contact manifold M and A is a tensor field of type (1,2), with $A_X Y = A(X,Y)$.

Now for any vector Y and for any tensor field f of type (1.1)

$$\begin{aligned}(\overline{\nabla}_X f)(Y) &= \overline{\nabla}_X(fY) - f(\overline{\nabla}_X Y)\\ &= \nabla_X(fY) + A_X fY - f\nabla_X Y - fA_X Y,\\ \overline{\nabla}_X f &= (\nabla_X f)Y + A_X \circ f - f \circ A_X.\end{aligned} \tag{2.9}$$

Now suppose that the distributions D^+, D^-, D^0 are parallel with respect to $\overline{\nabla}$, from proposition 1, 2 and 3, [3] we get

$$\overline{\nabla}_X \phi_1 \circ \phi_1 = 0, \qquad \overline{\nabla}_X \phi_2 \circ \phi_2 = 0, \qquad \overline{\nabla}_X \phi_3 \circ \phi_3 = 0.$$

Now by virtue of (2.8) and (2.9) we have

$$\begin{aligned}\nabla_X \phi_1 \circ \phi_1 + A_X \phi_1^2 - \phi_1 A_X \phi_1 &= 0,\\ \nabla_X \phi_2 \circ \phi_2 + A_X \phi_2^2 - \phi_2 A_X \phi_2 &= 0,\\ \nabla_X \phi_3 \circ \phi_3 + A_X \phi_3^2 - \phi_3 A_X \phi_3 &= 0.\end{aligned}$$

Now adding these equations and making use of (1.13) and (1.14) we get

$$\begin{aligned}\nabla_X \phi_1 \circ \phi_1 + \nabla_X \phi_2 \circ \phi_2 + \nabla_X \phi_3 \circ \phi_3 + A_X&\\ - \phi_1 A_X \phi_1 - \phi_2 A_X \phi_2 - \phi_3 A_X \phi_3 &= 0.\end{aligned}$$

Using (2.2) we get

$$\nabla_X \phi_1 \circ \phi_1 + \nabla_X \phi_2 \circ \phi_2 + \nabla_X \phi_3 \circ \phi_3 + PA_X = 0.$$

This equation is equivalent to

$$PA_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla \phi_2 + \phi_3 \nabla_X \phi_3.$$

Hence, in virtue of lemma 2, we obtain

$$A_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 + QS_X$$

where S_X is an orbitrary tensor field of type (1,2) with $S_X Y = S(X,Y)$.
Thus we have,

Theorem 3. *If, on an unified contact manifold M with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ there exists any linear connection Γ then distributions D^+, D^-, D^0 given by (1.7), (1.8), (1.9) respectively are parallel with respect to every connection Γ given by*

$$\overline{\nabla}_X = \nabla_X + \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 + QS_X$$

3. An unified Contact Riemannian manifold of P-Sasakian type [4]

Suppose that M is an unified contact Riemannian manifold with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. We define the following tensor fields of type (2.2).

$$F = \frac{1}{2}\{I\otimes I + \eta^\alpha\otimes\xi_\alpha\otimes I + I\otimes\eta^\alpha\otimes\xi_\alpha - \eta^\alpha\otimes\xi_\alpha\otimes\eta^\beta\otimes\xi_\beta - \frac{1}{a^2}\phi\otimes\phi\} \quad (3.1)$$

and

$$H = \frac{1}{2}\{I\otimes I - \eta^\alpha\otimes\xi_\alpha\otimes I - I\otimes\eta^\alpha\otimes\xi_\alpha + \eta^\alpha\otimes\xi_\alpha\otimes\eta^\beta\otimes\xi_\beta + \frac{1}{a^2}\phi\otimes\phi\} \quad (3.2)$$

with properties

$$F + H = I\otimes I, \qquad HH = H, \qquad FF = F, \qquad FH = HF = 0 \quad (3.3)$$

we introduce

$$L = H - \frac{1}{a^2}\phi\otimes\phi \quad (3.4)$$

Now we define a symmetric tensor field ϕ of type (0,2) as follows:

$$\phi(X,Y) = g(\phi X, Y)$$

Definition. An unified contact Riemannian manifold M, with a structure $\sum = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be of unified contact type of the following condition is satisfied:

$$2\phi(X,Y) = (\nabla_X\eta^\alpha)Y + (\nabla_Y\eta^\alpha)X \qquad \text{for all} \qquad \alpha\in(\mathrm{r}) \quad (3.5)$$

If, moreover, all η^α are closed, then since $d\eta^\alpha = 0$ is equivalent to $(\nabla_X\eta^\alpha)Y = (\nabla_Y\eta^\alpha)X$, the condition (3.5) is reduced to

$$\phi(X,Y) = (\nabla_X\eta^\alpha)Y, \qquad \text{for all} \qquad \alpha\in(\mathrm{r}) \quad (3.6)$$

From $\eta^\alpha(X) = g(X,\xi_\alpha)$, $\alpha\in(r)$ we have

$$(\nabla_X\eta^\alpha)Y = g(\nabla_X\xi_\alpha, Y)$$

and then (3.6) is equivalent to

$$\phi X = \nabla_X\xi_\alpha \qquad \text{for all} \qquad \alpha\in(\mathrm{r}) \quad (3.7)$$

now we prove the following.

Theorem 4. *Let M be an unified contact Riemannian manifold of unified contact type with structure $(\phi, \xi_\alpha, \eta^\alpha_{\alpha\in(r)}$. If*
(i) *all η^α are closed, and*
(ii) The tensor field ϕ satisfies the condition $L\nabla_Z\phi = 0$.
Then

$$\nabla_Z\phi(X,Y) = -a^2\sum_\alpha \eta^\alpha(X)[g(Y,Z) - \sum_\beta \eta^\beta(Y)\eta^\beta(Z)] - a^2\sum_\alpha \eta^\alpha(Y)[g(X,Z) - \sum_\beta \eta^\beta(X)\eta^\beta(Z)], \tag{3.8}$$

Proof. Since η^α are closed then (3.6) is satisfied, we have

$$\phi(\phi X, Y) = a^2[g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)], \tag{3.9}$$

$$\nabla_Z\phi(X,Y) = g((\nabla_Z\phi)X, Y), \tag{3.10}$$

From (3.9), we have

$$\nabla_Z\phi(\phi X, Y) = a^2\{\nabla_Z g(X,Y) - (\nabla_Z\eta^\alpha)(X)\sum_\alpha \eta^\alpha(Y) - (\nabla_Z\eta^\alpha)(Y)\sum_\alpha \eta^\alpha(X)\}, \tag{3.11}$$

Putting ϕ_Y instead of Y into (3.11), and making use of (3.6), we get

$$\nabla_Z\phi(\phi X, \phi Y) = a^2\{-\nabla_Z\phi(X,Y) - \sum_\alpha \eta^\alpha(Y)\,\phi(Z,X) - \sum_\alpha \eta^\alpha(X)\,\phi(Z,\phi Y)\} \tag{3.12}$$

From (3.6) and (3.10), we have

$$\nabla_Z\,\phi(X,\xi_\alpha) = g((\nabla_Z\,\phi(X,\xi_\alpha) = -g(\nabla_Z\xi_\alpha, \phi X) = -\phi(Z,\phi X) \tag{3.13}$$

and

$$\nabla_Z\,\phi(\xi_\alpha,\xi_\alpha) = 0 \tag{3.14}$$

The condition (ii) is equivalent

$$\begin{aligned} 0 &= (L\nabla_Z\,\phi)(X,Y) \\ &- \frac{1}{2}\{\nabla_Z\,\phi(X,Y) - \nabla_Z\,\phi(\eta^\alpha(X)\xi_\alpha, Y) - \nabla_Z\,\phi(X, \eta^\alpha(Y)\xi_\alpha) \\ &+ \nabla_Z\,\phi(\eta^\alpha(X)\xi_\alpha,\ \eta^\beta(Y)\xi_\beta) - \frac{1}{a^2}\nabla_Z\,\phi(\phi X, \phi Y)\} \\ &= \frac{1}{2}\{\nabla_Z\,\phi(X,Y) - \eta^\alpha(X)\nabla_Z\,\phi(\xi_\alpha, Y) - \eta^\alpha(Y)\nabla_Z\,\phi(X,\xi_\alpha) \\ &+ \eta^\alpha(X)\eta^\beta(Y)\nabla_Z\,\phi(\xi_\alpha,\xi_\beta) - \frac{1}{a^2}\nabla_Z\,\phi(\phi X, \phi Y)\} \end{aligned}$$

On account of (3.12), (3.13) and (3.14) we have

$$\nabla_Z\,\phi(X,Y) \;=\; -\sum_\alpha \eta^\alpha(Y)\,\phi(Z,\phi X) \;-\; \sum_\alpha \eta^\alpha(X)\,\phi(Z,\phi Y) \tag{3.15}$$

Now using (3.9) in (3.15), we get

$$\begin{aligned}\nabla_Z\,\phi(X,Y) \;=\; & -a^2\sum_\alpha \eta^\alpha(Y)\,[g(X,Z) \;-\; \sum_\beta \eta^\beta(X)\,\eta^\beta(Z)] \\ & -a^2\sum_\alpha \eta^\alpha(X)\,[g(Y,Z) \;-\; \sum_\beta \eta^\beta(Y)\eta^\beta(Z)]\end{aligned}$$

Hence the theorem.

Definition. An unifined contact Riemannian manifold M, with structure

$$\sum \;=\; (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$$

satisfying the conditions (3.6) and (3.8) is said to be of P-Sasakian type.

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