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ON SUBCLASSES OF P-VALENT CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $K[C,D,p,\alpha]$, $-1 \le D < C \le 1$ and $0 \le \alpha < p$ denote the class of functions $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ analytic in the unit disc $U = \{z : |z| < 1\}$ and satisfying the condition $1 + \frac{zg''(z)}{g'(z)}$ is subordinate to $\frac{p + [pD + (C-D)(p-\alpha)]z}{1+Dz}$, $z \in U$. We investigate the subclass of p-valent close-to-convex functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$, for which there exists $g(z) \in K[C,D,p,\alpha]$ such that $\frac{pf'(z)}{g'(z)}$ is subordinate to $\frac{p + [pB + (A-B)(p-\beta)]z}{1+Bz}$, $-1 \le B < A \le 1$ and $0 \le \beta < p$. Distortion and rotation theorems and coefficient bounds are obtained.

1. Introduction

Let A_p (p a fixed integer greater than zero) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in $U = \{z : |z| < 1\}$. Let Ω denote the class of bounded analytic functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| \le |z|$ for $z \in U$.

For $-1 \le B < A \le 1$and $0 \le \beta < p$, denote by $P[A,B,p,\beta]$ the class of functions $p(z) = p + \sum_{k=1}^{\infty} c_k z^k$ which are analytic in U and which satisfy that $p(z) \in P[A,B,p,\beta]$ if and only if

$$p(z) \prec \frac{p + [pB + (A-B)(p-\beta)]z}{1+Bz}, \qquad z \in U.$$

By definition of subordination if follows that $p(z) \in P[A,B,p,\beta]$ has a representation of the form

$$p(z) = \frac{p + [pB + (A-B)(p-\beta)]w(z)}{1+Bw(z)}, \qquad w \in \Omega. \tag{1.1}$$

The class $P[A,B,p,\beta]$ was introduced by Aouf [1].

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Given C, D, $-1 \leq D < C \leq 1$ and $0 \leq \alpha < p$, $K[C,D,p,\alpha]$ and $P^*[C,D,p,\alpha]$ denote the classes of functions $f(z) \in A_p$ such that $1 + \frac{zf''(z)}{f'(z)} \in P[C,D,p,\alpha]$ and $\frac{zf'(z)}{f(z)} \in P[C,D,p,\alpha]$, respectively. The class $P^*[C,D,p,\alpha]$ was introduced by Aouf [1].

It follows from the definitions of the classes $K[C,D,p,\alpha]$ and $P^*[C,D,p,\alpha]$ that

$$g(z) \in K[C,D,p,\alpha] \quad \text{if and only if} \quad \frac{zg'(z)}{p} \in P^*[C,D,p,\alpha]. \tag{1.2}$$

We note that:

1. $P^*[1,-1,1,\alpha] = S^*(\alpha)$, $K[1,-1,1,\alpha] = C(\alpha)$, $0 \leq \alpha < 1$, are the well-known classes of starlike functions of order α and convex functions of order α, respectively, introduced by Robertson [15].

2. $P^*[1,-1,p,\alpha] = S_p^*(\alpha)$ and $K[1,-1,p,\alpha] = C_p(\alpha)$, $0 \leq \alpha < p$, are, respectively, the class of p-valent starlike functions of order α, investigated by Goluzina [6] and the class of p-valent convex functions of order α.

3. $P^*[C,D,1,0] = P^*[C,D]$, is the class of functions $f(z) \in A_1$, introduced by Janowski [7] and studied further by Goel and Mehrok [3,4] and $K[C,D,1,0] = K[C,D]$, is the class of functions $f(z) \in A_1$, studied by Mazur [13] and Silvia [19].

A function $f(z) \in A_p$ is said to be in the class $C[A,B;C,D,p,\beta,\alpha]$, $-1 \leq A < B \leq 1$, $-1 \leq D < C \leq 1$, $0 \leq \beta < p$ and $0 \leq \alpha < p$, if there exists $g(z) \in K[C,D,p,\alpha]$ such that

$$\frac{pf'(z)}{g'(z)} \in P[A,B,p,\beta]. \tag{1.3}$$

We note that:

1. $C[1,-1;1,-1,1,0,0] = C$, is the well-known class of close-to-convex functions, introduced by Kaplan [8].

2. $C[A,B,;C,D,1,0,0] = C[A,B;C,D]$, is the class of functions $f(z) \in A_1$, studied by Silvia [19].

3. $C[1,-1;C,D,1,0,0]$, was studied by Goel and Mehrok [4,5].

4. $C[1,-1;1,-1,1,\beta,\alpha] = C(\alpha,\beta)$, is the class of close-to-convex functions of order α and type β, was introduced by Libera [11].

5. In [2] Aouf studied the class $C[1,-1;C,D,p,\beta,\alpha] = C[C,D,p,\beta,\alpha]$ of functions $f(z) \in A_p$ satisfying

$$\frac{zf'(z)}{g(z)} \in P[1,-1,p,\beta] = P(p,\beta),$$
$$g(z) \in P^*[C,D,p,\alpha].$$

2. Distortion And Rotation Theorems.

Unless otherwise mentioned in the sequel, the only restrictions on the real constants $A, B, C, D, \alpha, \beta$ and p are that $-1 \leq D < C \leq 1$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < p$ and $0 \leq \beta < p$.

Theorem 1. *For $f(z) \in C[A, B; C, D, p, \beta, \alpha]$, $|z| \leq r < 1$,*

$$r^{p-1}\frac{p-[pB+(A-B)(p-\beta)]r}{1-Br}(1-Dr)^{(\frac{C-D}{D})(p-\alpha)} \leq |f'(z)|$$

$$\leq r^{p-1}\frac{p+[pB+(A-B)(p-\beta)]r}{1+Br}(1+Dr)^{(\frac{C-D}{D})(p-\alpha)}, \quad D \neq 0,$$

$$r^{p-1}\frac{p-[pB+(A-B)(p-\beta)]r}{1-Br}e^{-C(p-\alpha)r} \leq |f'(z)|$$

$$\leq r^{p-1}\frac{p+[pB+(A-B)(p-\beta)]r}{1+Br}e^{C(p-\alpha)r}, \quad D = 0.$$

The bounds are sharp.

Proof. For $f(z) \in C[A, B; C, D, p, \beta, \alpha]$, there exists a $g(z) \in K[C, D, p, \alpha]$ and $p(z) \in P[A, B, p, \beta]$ such that

$$f'(z) = \frac{g'(z)}{p}p(z). \tag{2.1}$$

Since $g(z) \in K[C, D, p, \alpha]$ if and only if $\frac{zg'(z)}{p} \in P^*[A, B, p, \alpha]$, for $|z| \leq r < 1$ [1, Theorem 1]

$$pr^{p-1}(1-Dr)^{(\frac{C-D}{D})(p-\alpha)} \leq |g'(z)| \leq pr^{p-1}(1+Dr)^{(\frac{C-D}{D})(p-\alpha)}, \quad D \neq 0,$$

and

$$pr^{p-1}e^{-C(p-\alpha)r} \leq |g'(z)| \leq pr^{p-1}e^{C(p-\alpha)r}, \quad D = 0. \tag{2.2}$$

Also for $p(z) \in P[A, B, p, \beta]$, we have for $|z| \leq r < 1$ [1, Corollary 1]

$$\frac{p-[pB+(A-B)(p-\beta)]r}{1-Br} \leq |p(z)| \leq \frac{p+[pB+(A-B)(p-\beta)]r}{1+Br}. \tag{2.3}$$

The result follows immediately upon applying (2.3) and (2.2) to (2.1).

Equality is obtained for $f(z) \in C[A, B; C, D, p, \beta, \alpha]$ satisfying

$$f'(z) = \begin{cases} z^{p-1}(1+Dz)^{(\frac{C-D}{D})(p-\alpha)} \cdot \dfrac{p+[pB+(A-B)(p-\beta)]z}{1+Bz}, & D \neq 0, \\ z^{p-1}e^{C(p-\alpha)z} \cdot \dfrac{p+[pB+(A-B)(p-\beta)]z}{1+Bz}, & D = 0 \end{cases} \tag{2.4}$$

and $z = \pm r$.

Remarks.

1. For $p = 1$ and $\alpha = \beta = 0$, Theorem 1 agrees with Theorem 1 of Silvia [19].

2. For $A = 1$, $B = -1$, $p = 1$ and $\alpha = \beta = 0$, Theorem 1 agrees with Theorem 3 of Goel and Mehrok [3].

3. For $A = 1$ and $B = -1$, Theorem 1 agrees with Theorem 2 of Aouf [2].

Theorem 2. *For $f(z) \in C[A,B;C,D,p,\beta,\alpha]$, $|z| \le r < 1$,*

$$
|\arg \frac{f'(z)}{z^{p-1}}| \le
\begin{cases}
(\frac{C-D}{D})(p-\alpha)\sin^{-1}(Dr) + \sin^{-1}\frac{(A-B)(p-\beta)r}{p-[pB+(A-B)(p-\beta)]Br^2}, & D \ne 0,\\
C(p-\alpha)r + \sin^{-1}\frac{(A-B)(p-\beta)r}{p-[pB+(A-B)(p-\beta)]Br^2}, & D = 0.
\end{cases}
$$

These inequalities are sharp.

Proof. From (2.1) and (1.2), we have

$$
|\arg \frac{f'(z)}{z^{p-1}}| \le |\arg \frac{f_1(z)}{z^p}| + |\arg p(z)|, \tag{2.5}
$$

$f_1(z) \in P^*[C,D,p,\alpha]$ and $p(z) \in P[A,B,p,\beta]$.
For $f_1(z) \in P^*[C,D,p,\alpha]$, we know [1, Theorem 2] that for $|z| \le r < 1$

$$
|\arg \frac{f_1(z)}{z^p}| \le
\begin{cases}
(\frac{C-D}{D})(p-\alpha)\sin^{-1}(Dr), & D \ne 0,\\
C(p-\alpha)r, & D = 0.
\end{cases} \tag{2.6}
$$

Also for $p(z) \in P[A,B,p,\beta]$, we know [1, Theorem 4] that for $|z| \le r < 1$

$$
|\arg p(z)| \le \sin^{-1}\frac{(A-B)(p-\beta)r}{p-[pB+(A-B)(p-\beta)]Br^2}. \tag{2.7}
$$

Substituting (2.6) and (2.7) into (2.5) gives the result.

Equality is attained for $f(z) \in C[A,B;C,D,p,\beta,\alpha]$ satisfying

$$
f'(z) =
\begin{cases}
pz^{p-1}\frac{1+[B+(A-B)(1-\frac{\beta}{p})]\delta_1 z}{1+B\delta_1 z}(1+D\delta_2 z)^{(\frac{C-D}{D})(p-\alpha)}, & D \ne 0,\\
pz^{p-1}\frac{1+[B+(A-B)(1-\frac{\beta}{p})]\delta_1 z}{1+B\delta_1 z}e^{C(p-\alpha)\delta_2 z}, & D = 0,
\end{cases}
$$

where

$$\delta_1 = \frac{r}{z}\Big\{\frac{-([B+(A-B)(1-\frac{\beta}{p})]+B)r}{1+[B+(A-B)(1-\frac{\beta}{p})]Br^2}$$

$$+i\frac{\sqrt{1-[B+(A-B)(1-\frac{\beta}{p})]^2r^2}\sqrt{1-B^2r^2}}{1+[B+(A-B)(1-\frac{\beta}{p})]Br^2}\Big\},\ r = |z|$$

and

$$\delta_2 = \frac{r}{z}\cdot[-Dr + i\sqrt{1-D^2r^2}].$$

Remarks.

1. For $p=1$ and $\alpha=\beta=0$, Theorem 2 agrees with Theorem 2 of Silvia [19].
2. For $A=1,\ B=-1,\ p=1$ and $\alpha=\beta=0$, Theorem 2 agrees with Theorem 4 of Goel and Mehrok [4].
3. For $A=1$ and $B=-1$, Theorem 2 agrees with Theorem 3 of Aouf [2].
4. For $p=A=C=1$ and $B=D=-1$, Theorem 2, agrees with Theorem 4 of Silverman [18].
5. Choosing $p=A=C=1,\ B=D=-1$ and $\alpha=\beta=0$ in Theorem 2, we get the result due to Ogawa [14] and Krzyz [10].

3. Hadamard product.

The convolution or Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series $(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. In order to obtain a subordination result linking $C[A,B;C,D,p,\beta,\alpha]$ and $P[A,B,p,\beta]$ we need the following lemma:

Lemma 1 (Ruscheweyh and Sheil-Small, [17]). *If $\psi(z)$ is regular in U, $\phi(z)$ and $h(z)$ are convex univalent in U such that $\psi(z) \prec \phi(z)$, then $\psi(z)*h(z) \prec \phi(z)*h(z),\ z\in U$.*

Theorem 3. *If $f(z)\in C[A,B;C,D,p,\beta,\alpha]$, then there exists $p(z)\in P[A,B,p,\beta]$ such that for all s and t with $|s|\le 1,\ |t|\le 1\ (s\neq t)$,*

$$\frac{f'(sz)p(tz)t^{p-1}}{f'(tz)p(sz)s^{p-1}}$$

$$\prec \begin{cases} \left(\frac{1+Dsz}{1+Dtz}\right)^{\left(\frac{C-D}{D}\right)(p-\alpha)} & ,D \neq 0, \quad (3.1)\\ e^{C(p-\alpha)(s-t)z} & ,D = 0. \quad (3.2)\end{cases}$$

Proof. The proof is similar to the one given by Ruscheweyh [16], Goel and Mehrok [4] and Silvia [19].

We first consider the case when $D \neq 0$. We have

$$\frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z)} - p + 1 = (1 + \frac{zg''(z)}{g'(z)}) - p,$$

$g(z) \in K[C, D, p, \alpha]$ and $p(z) \in P[A, B, p, \beta]$. Therefore,

$$\frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z)} - p + 1 \prec \frac{(C-D)(p-\alpha)z}{1+Dz}, \tag{3.3}$$

where $\frac{(C-D)(p-\alpha)z}{1+Dz}$ is convex, univalent in U. For $|s| \leq 1$, $|t| \leq 1$, $(s \neq t)$,

$$h(z) = \int_0^z (\frac{s}{1-su} - \frac{t}{1-tu})du \tag{3.4}$$

is convex, univalent in U. (3.3) and (3.4) satisfy the conditions of Lemma 1, and therefore

$$(\frac{zf''(z)}{f'(z)} - \frac{zp'(z)}{p(z)} - p + 1) * h(z) \prec \frac{(C-D)(p-\alpha)z}{1+Dz} * h(z). \tag{3.5}$$

Given any function $\ell(z)$ analytic in U with $\ell(0) = 0$, we have

$$(\ell * h)(z) = \int_{tz}^{sz} \ell(u)\frac{du}{u}, \qquad z \in U. \tag{3.6}$$

By the application of (3.6) and (3.5) can be written as

$$\int_{tz}^{sz} [\frac{uf''(u)}{f'(u)} - \frac{up'(u)}{p(u)} - p + 1]\frac{du}{u} \prec (C-D)(p-\alpha)\int_{tz}^{sz} \frac{du}{1+Du}$$

from which (3.1) follows.
Similarly for $D = 0$, we obtain (3.2).

Corollary 1. *If $f(z) \in C[A, B; C, D, p, \beta, \alpha]$, then there exists a $p(z) \in P[A, B, p, \beta]$ and a Schwarz function $w(z) \in \Omega$ such that*

$$\frac{f'(z)}{z^{p-1}} = \begin{cases} p(z)(1 + Dw(z))^{(\frac{C-D}{D})(p-\alpha)}, & D \neq 0, \\ p(z)e^{C(p-\alpha)w(z)}, & D = 0. \end{cases}$$

Proof. The result follows directly upon substituting $s = 1$ and $t = 0$ into Theorem 3.

Corollary 2. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C[A, B; C, D, p, \beta, \alpha]$, *then*

$$| a_{p+1} | \leq \frac{p(C-D)(p-\alpha) + (A-B)(p-\beta)}{p+1}.$$

Proof. If $g \prec F$, then $|g'(0)| \leq |F'(0)|$ [12]. From Corollary 1, we take $g(z) = f'(z)/z^{p-1}p(z)$ and

$$F(z) = \begin{cases} (1+Dz)^{(\frac{C-D}{D})(p-\alpha)} & , D \neq 0, \\ e^{C(p-\alpha)z} & , D = 0. \end{cases}$$

Then $g'(0) = \dfrac{(p+1)a_{p+1} - c_1}{p}$ for $p(z) = p + \sum_{n=1}^{\infty} c_n z^n$ and $F'(0) = (C-D)(p-\alpha)$. Therefore $\dfrac{(p+1)|a_{p+1}| - |c_1|}{p} \leq |(C-D)(p-\alpha)|$

and

$$\begin{aligned} | a_{p+1} | &\leq \frac{p\,|\,(C-D)(p-\alpha)\,| + |\,c_1\,|}{p+1} \\ &\leq \frac{p(C-D)(p-\alpha) + (A-B)(p-\beta)}{p+1} \qquad [1, \text{ Theorem } 5] \end{aligned}$$

as claimed.

4. Coefficient Inequalities.

We begin with coefficient inequalities for $K[C, D, p, \alpha]$.

Lemma 2. *For* $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in K[C, D, p, \alpha]$ *and* μ *complex*

$$| b_{p+1} | \leq (\frac{p}{p+1})(C-D)(p-\alpha), \tag{4.1}$$

and

$$\begin{aligned} | b_{p+2} - \mu b_{p+1}^2 | \leq \; & \frac{p}{2(p+2)}(C-D)(p-\alpha) \cdot \\ & \cdot \max \{1, | \frac{2p(p+2)}{(p+1)^2}\mu(C-D)(p-\alpha) - [(C-D)(p-\alpha)p - D] | \}. \end{aligned} \tag{4.2}$$

The result is sharp.

Proof. For $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in K[C, D, p, \alpha]$, there exists a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n \in \Omega$ such that

$$1 + \frac{zg''(z)}{g'(z)} = \frac{p + [pD + (C-D)(p-\alpha)]w(z)}{1 + Dw(z)}$$

or

$$1 + \frac{zg''(z)}{g'(z)} - p = \frac{(C-D)(p-\alpha)w(z)}{1 + Dw(z)}.$$

Substituting of the series expansions and comparison of the coefficients leads to

$$b_{p+1} = (\frac{p}{p+1})(C-D)(p-\alpha)\gamma_1$$

and

$$b_{p+2} = \frac{p}{2(p+2)}(C-D)(p-\alpha)\{\gamma_2 + [(C-D)(p-\alpha)p - D]\gamma_1^2\}.$$

Therefore,

$$| b_{p+1} | \leq (\frac{p}{p+1})(C-D)(p-\alpha) \tag{4.3}$$

and

$$\begin{aligned} b_{p+2} - \mu b_{p+1}^2 = & \frac{p}{2(p+2)}(C-D)(p-\alpha)\{\gamma_2 + [(C-D)(p-\alpha)p - D) \\ & - \frac{2p(p+2)}{(p+1)^2}\mu(C-D)(p-\alpha)]\gamma_1^2\}. \end{aligned} \tag{4.4}$$

We know [9] that for $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n \in \Omega$, if s is any complex number, then

$$| \gamma_2 - s\gamma_1^2 | \leq \max\{1, | s |\}. \tag{4.5}$$

Equality is attained for $w(z) = z^2$ and $w(z) = z$.
Combining (4.4) and (4.5) yields the result, and since (4.5) is sharp, then (4.2) is also sharp.

Theorem 4. *If* $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in K[C, D, p, \alpha]$, *then*

$$| b_n | \leq \frac{p}{n(n-p)!} \prod_{k=0}^{n-(p+1)} | (D-C)(p-\alpha) + Dk | \tag{4.6}$$

for $n \geq p+1$, *and these bounds are sharp for all admissible* C, D *and* α *and for each* n.

Proof. If $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in K[C, D, p, \alpha]$, then $\frac{zg'(z)}{p} = z^p + \sum_{n=p+1}^{\infty} \frac{n}{p} b_n z^n$ is in $P^*[C, D, p, \alpha]$. But for $f_1(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in P^*[C, D, p, \alpha]$, we have [1, Theorem 3].

$$| a_n | \leq \frac{1}{(n-p)!} \prod_{k=0}^{n-(p+1)} | (D-C)(p-\alpha) + Dk | . \tag{4.7}$$

Then the result follows from (4.7) and replacing a_n by $\frac{n}{p} b_n$.

For sharpness of (4.6) consider

$$g'(z) = pz^{p-1}(1 - D\delta z)^{(\frac{C-D}{D})(p-\alpha)}, \ | \delta | = 1, \ D \neq 0. \tag{4.8}$$

Theorem 5. *For* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C[A, B; C, D, p, \beta, \alpha]$

$$| a_{p+1} | \leq \frac{p(C-D)(p-\alpha) + (A-B)(p-\beta)}{p+1} \tag{4.9}$$

and

$$| a_{p+2} | \leq \begin{cases} \frac{p}{2(p+2)}(C-D)(p-\alpha) + \frac{(A-B)(p-\beta)}{p+2}[(C-D)(p-\alpha) + 1] \\ + \frac{p}{p+2}(C-D)^2(p-\alpha)^2 + \frac{p}{p+2}B^2, \ | (C-D)(p-\alpha)p - D | \leq 1, & (4.10) \\ \\ \frac{p}{2(p+2)}(C-D)(p-\alpha)[(C-D)(p-\alpha)p - D] \\ + \frac{(A-B)(p-\beta)[(C-D)(p-\alpha) + 1]}{p+2} + \frac{p}{p+2}(C-D)^2(p-\alpha)^2 & (4.11) \\ + \frac{p}{p+2}B^2, \ | (C-D)(p-\alpha)p - D | > 1. \end{cases}$$

Proof. There exists a $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in K[C, D, p, \alpha]$ and a Schwarz function $w(z) = \sum_{n=1}^{\infty} \gamma_n z^n \in \Omega$ such that

$$\frac{pf'(z)}{g'(z)} = \frac{p + [pB + (A-B)(p-\beta)]w(z)}{1 + Bw(z)}, \ z \in U.$$

Comparing series expansions, we see

$$a_{p+1} = b_{p+1} + \frac{(A-B)(p-\beta)}{p+1}\gamma_1$$

and

$$\begin{aligned} a_{p+2} &= b_{p+2} + \frac{p+1}{p(p+2)}(A-B)(p-\beta)b_{p+1}\gamma_1 \\ &+ \frac{(A-B)(p-\beta)}{p+2}\{\gamma_2 - B\gamma_1^2\} + \frac{(p+1)^2}{p(p+2)}b_{p+1}^2 - \frac{p}{p+2}B^2\gamma_1^2. \quad (4.12) \end{aligned}$$

The bound for $|a_{p+1}|$ follows from Lemma 2. Applying (4.5) and Lemma 2 ($\mu = 0$) to (4.12), we have

$$\begin{aligned} | a_{p+2} | &\le \frac{p}{2(p+2)}(C-D)(p-\alpha)\ \max\{1,\ |(C-D)(p-\alpha)p - D|\} \\ &+ \frac{(A-B)(p-\beta)(C-D)(p-\alpha)}{p+2} + \frac{(A-B)(p-\beta)}{p+2}\max\{1, |B|\} \\ &+ \frac{p}{p+2}(C-D)^2(p-\alpha)^2 + \frac{p}{p+2}B^2 \\ &= \frac{p}{2(p+2)}(C-D)(p-\alpha)\ \max\{1, |(C-D)(p-\alpha)p - D|\} \\ &+ \frac{(A-B)(p-\beta)}{p+2}[(C-D)(p-\alpha)+1] + \frac{p}{p+2}(C-D)^2(p-\alpha)^2 \\ &+ \frac{p}{p+2}B^2. \end{aligned}$$

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