TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 2, Summer 1991

ON LIEB AND THIRRING TYPE DISCRETE INEQUALITIES

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Abstract. Discrete inequalities of the Lieb and Thirring type involving functions of several independent variables and their forward differences are established. The proofs given here are elementary and the results established provide new estimates on these types of inequalities.

1. Introduction

In [4] Lieb and Thirring have given the following interesting inequality.

Let $u_r (r = 1, \cdots, M)$ be a finite family of functions in $H^1(R^n)$ which are orthonormal in $L^2(R^n)$ and let p be a constant satisfying $\max(1, \frac{n}{2}) < p \leq 1 + (\frac{n}{2})$. Then

$$\left[\int_{R^n} \{\sum_{r=1}^{M} u_r(x)^2\}^{\frac{p}{p-1}} dx\right]^{\frac{2(p-1)}{n}} \leq k_0 \sum_{r=1}^{M} \int_{R^n} \sum_{i=1}^{n} \{\frac{\partial}{\partial x_i} u_r(x)\}^2 dx, \qquad (1)$$

where $k_0 = k_0(n,p)$ is the constant independent of M and u_r.

The inequality (1) is an extension of the well known Sobolev-Gagliardo-Nirenberg inequality (see, [2, 5,8]). A number of interesting generalizations of the inequality (1) which have applications in the study of the dimension of attractors associated with dissipative parabolic equations are recently given by Ghidaglia, Marion and Temam in [3]. Aside from the applications, inequalities like (1) are of interest in their own right and we believe that the discrete inequalities of the type (1) will be a new addition to the literature on such inequalities. The main purpose of the present paper is to establish some new discrete inequalities of the type (1) involving functions of several independent variables and their forward differences. The method used in the proofs is elementary and our results provide new estimates on these types of discrete inequalities.

2. Statement of results

In what follows, we let R be the set of real numbers and $N = \{1, 2, \cdots\}$. For $x = (x_1, \ldots, x_n) \in N^n$ and $z(x) : N^n \to R$, we define the forward difference operators as

Received August 23, 1989, revised February 22, 1990.

follows:

$$\Delta_1 z(x_1,\dots,x_n) = z(x_1+1,x_2,\dots,x_n) - z(x_1,\dots,x_n),$$

$$\vdots$$

$$\Delta_n z(x_1,\dots,x_n) = z(x_1,\dots,x_{n-1},x_n+1) - z(x_1,\dots,x_n).$$

The notation $\Delta_i z(x_1,\cdots,y_i,\cdots,x_n)$ for $i=1,\cdots,n$ we mean for $i=1$ it is $\Delta_1 z(y_1,x_2,\dots,x_n)$ and so on and for $i=n$ it is $\Delta_n z(x_1,\dots,x_{n-1},y_n)$. Let B be a bounded domain in N^n with $n\geq 1$ defined by $B=\{x:\underline{1}\leq x\leq a+\underline{1}\}$ where $\underline{1}=(1,\cdots,1)\in N^n$, $x=(x_1,\dots,x_n)\in N^n$, $a=(a_1,\dots,a_n)\in N^n$. We define $\alpha=\max\{a_1,\dots,a_n\}$. We denote by $F(B)$ the class of functions $z(x):B\to R$ for which $\Delta_i z(x_1,\dots,y_i,\dots,x_n)$ exist and such that

$$z(1,x_2,\dots,x_n)=z(x_1,1,x_3,\dots,x_n)=\cdots=z(x_1,\dots,x_{n-1},1)=0,$$
$$z(a_1+1,x_2,\dots,x_n)=z(x_1,a_2+1,x_3,\dots,x_n)=\cdots=z(x_1,\dots,x_{n-1},a_n+1)=0$$

For $z(x):B\to R$ we use the following notation

$$\sum_B z(y) = \sum_{y_1=1}^{a_1}\cdots\sum_{y_n=1}^{a_n} z(y_1,\dots,y_n),$$

and also use the customary convention

$$\sum_{y_1=a_1}^{a_1-1} z(y_1,x_2,\dots,x_n) = 0,\ \cdots,\ \sum_{y_n=a_n}^{a_n-1} z(x_1,\dots,x_{n-1},y_n) = 0.$$

Our main result is established in the following theorem.

Theorem 1. *Let $u_r\in F(B)$ for $r=1,\cdots,M$ and let $m\geq 1$ and $p\geq 2$ be real constants. Then*

$$\Big[\sum_B\Big\{\sum_{r=1}^M |u_r(y)|^2\Big\}^{\frac{p}{p-1}}\Big]^{\frac{2m(p-1)}{p}} \leq k_1\sum_{r=1}^M\sum_B\sum_{i=1}^n |\Delta_i u_r(y)|^{4m}, \tag{2}$$

where

$$k_1 = \frac{1}{n}\Big(\frac{1}{4}\Big)^{2m} M^{2m-1}\alpha^{\frac{p(4m+2nm-n)-2nm}{p}}$$

Remark 1. If we take $m=1$ in (2), then we get the inequality analogous to the discrete version of the inequality (1). On taking $u_r(x)=u(x)$ for $r=1,\cdots,M$ and $m=1$, $p=2$ in (2) we get the following inequality

$$\sum_B |u(x)|^4 \leq \frac{1}{n}\Big(\frac{\alpha}{2}\Big)^4\sum_B\sum_{i=1}^n |\Delta_i u(y)|^4. \tag{3}$$

For different versions of inequalities of the type (2)-(3), see [6,7].

A slight variant of the inequality (2) motivated by the Dubinskii's inequalities [1] is embodied in the following theorem.

Theorem 2. *Let u_r, m, p be as defined in Theorem 1. Then*

$$\Big[\sum_B \Big\{\sum_{r=1}^{M} \mid u_r(y) \mid^2 \Big\}^{\frac{p}{p-1}}\Big]^{\frac{2m(p-1)}{p}}$$

$$\le k_2 \sum_{r=1}^{M}\sum_B\sum_{i=1}^{n} \mid \Delta_i u_r(y) \mid^{4m} + k_3 \sum_{r=1}^{M}\sum_B\sum_{i=1}^{n} \mid u_r(y) \mid^{2m} \mid \Delta_i u_r(y) \mid^{2m}, \qquad (4)$$

where

$$k_2 = \frac{1}{2n} M^{2m-1} \alpha^{\frac{p(2m+2nm-n)-2nm}{p}}, \qquad k_3 = 2^{2m} k_2.$$

Remark 2. In the special case when $m = 1$, $p = 2$ and $u_r(x) = u(x)$ for $r = 1, \cdots, M$, inequality (4) reduces to

$$\sum_B \mid u(y) \mid^4 \le \frac{\alpha^2}{2n} \sum_B \sum_{i=1}^{n} \mid \Delta_i u(y) \mid^4 + \frac{2\alpha^2}{n} \sum_B \sum_{i=1}^{n} \mid u(y) \mid^2 \mid \Delta_i u(y) \mid^2. \qquad (5)$$

It is interesting to note that the inequality (5) is analogous to the discrete analogue of the Sobolev's inequality [2]. However the bound obtained in (5) is not exactly the discrete analogue of the bound involved on the right side of the Sobolev's inequality (see, [5,p.1]).

3. Proofs of Theorems 1 and 2

Since $u_r \in F(B)$, we have the following identities:

$$n u_r(x) = \sum_{i=1}^{n} \Big\{ \sum_{y_i=1}^{x_i-1} \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) \Big\}, \qquad (6)$$

$$n u_r(x) = -\sum_{i=1}^{n} \Big\{ \sum_{y_i=x_i}^{a_i} \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) \Big\}, \qquad (7)$$

for $r = 1, \cdots, M$. From (6) and (7) we obtain

$$\mid u_r(x) \mid \le \frac{1}{2n} \sum_{i=1}^{n} \Big\{ \sum_{y_i=1}^{a_i} \mid \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) \mid \Big\}. \qquad (8)$$

From (8) and on using the elementary inequality (see, [5, p.5])

$$\Big\{\sum_{i=1}^{n} c_i\Big\}^k \le n^{k-1} \sum_{i=1}^{n} c_i^k, \qquad (9)$$

where $c_1, \ldots, c_n \geq 0$ reals $k \geq 1$, Schwarz inequality and the definition of α, we obtain

$$
\begin{aligned}
| u_r(x) |^2 &\leq (\frac{1}{2n})^2 \Big[\sum_{i=1}^{n} \{ \sum_{y_i=1}^{a_i} | \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) | \} \Big]^2 \\
&\leq (\frac{1}{2n})^2 n \sum_{i=1}^{n} \{ \sum_{y_i=1}^{a_i} | \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) | \}^2 \\
&\leq (\frac{\alpha}{4n}) \sum_{i=1}^{n} \{ \sum_{y_i=1}^{a_i} | \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) |^2 \}. \qquad (10)
\end{aligned}
$$

From (10) and on using inequality (9) repeatedly, Hölder's inequality with indices p, $\frac{p}{p-1}$ and the definition of α, we obtain

$$
\begin{aligned}
\{ \sum_{r=1}^{M} | u_r(x) |^2 \}^{\frac{p}{p-1}} &\leq (\frac{\alpha}{4n})^{\frac{p}{p-1}} (mn)^{\frac{p}{p-1} - 1} \qquad (11) \\
&\quad \cdot \sum_{r=1}^{M} \{ \sum_{i=1}^{n} \{ \sum_{y_i=1}^{a_i} | \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) |^2 \}^{\frac{p}{p-1}} \} \\
&\leq (\frac{\alpha}{4n})^{\frac{p}{p-1}} (Mn)^{\frac{1}{p-1}} \alpha^{\frac{1}{p-1}} \\
&\quad \cdot \sum_{r=1}^{M} \{ \sum_{i=1}^{n} \{ \sum_{y_i=1}^{a_i} | \Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n) |^{\frac{2p}{p-1}} \} \}.
\end{aligned}
$$

Setting $x_i = y_i$ $(i = 1, \cdots, n)$ in (11) and taking the sum over both sides of (11) with respect to $y_1, \ldots, y_n$ on B and using the definition of α, we have

$$
\begin{aligned}
\sum_{B} \{ \sum_{r=1}^{M} | u_r(y) |^2 \}^{\frac{p}{p-1}} &\leq (\frac{\alpha}{4n})^{\frac{p}{p-1}} (Mn\alpha)^{\frac{1}{p-1}} \alpha \\
&\quad \cdot \sum_{r=1}^{M} \{ \sum_{i=1}^{n} \{ \sum_{B} | \Delta_i u_r(y) |^{\frac{2p}{p-1}} \} \}. \qquad (12)
\end{aligned}
$$

From (12) and on using the inequality (9) repeatedly, Hölder's inequality with indices $\frac{2m(p-1)}{p}$, $\frac{2m(p-1)}{2m(p-1) - p}$ and the definition of α, we obtain

$$
\Big[\sum_{B} \{ \sum_{r=1}^{M} | u_r(y) |^2 \}^{\frac{p}{p-1}} \Big]^{\frac{2m(p-1)}{p}}
$$

$$\le \{(\frac{\alpha}{4n})^{\frac{p}{p-1}}(Mn\alpha)^{\frac{1}{p-1}}\alpha\}^{\frac{2m(p-1)}{p}}(Mn)^{\frac{2m(p-1)}{p}-1}$$

$$\times \sum_{r=1}^{M}\Big\{\sum_{i=1}^{n}\{\sum_{B}|\Delta_i u_r(y)|^{\frac{2p}{p-1}}\}^{\frac{2m(p-1)}{p}}\Big\}$$

$$\le \{(\frac{\alpha}{4n})^{\frac{p}{p-1}}(Mn\alpha)^{\frac{1}{p-1}}\alpha\}^{\frac{2m(p-1)}{p}}(Mn)^{\frac{2m(p-1)}{p}-1}$$

$$\times (\alpha^n)^{\frac{2m(p-1)-p}{p}}\sum_{r=1}^{M}\Big\{\sum_{i=1}^{n}\{\sum_{B}|\Delta_i u_r(y)|^{4m}\}\Big\}$$

$$= K_1\sum_{r=1}^{M}\sum_{B}\sum_{i=1}^{n}|\Delta_i u_r(y)|^{4m}. \tag{13}$$

This completes the proof of Theorem 1.

From the assumptions on the functions $u_r(x)$ in Theorem 2, we have the following identities:

$$nu_r^2(x) = \sum_{i=1}^{n}\{\sum_{y_i=1}^{x_i-1}\Delta_i u_r^2(x_1,\ldots,y_i,\ldots,x_n)\}, \tag{14}$$

$$nu_r^2(x) = -\sum_{i=1}^{n}\{\sum_{y_i=x_i}^{a_i}\Delta_i u_r^2(x_1,\ldots,y_i,\ldots,x_n)\}, \tag{15}$$

for $r = 1,\cdots,M$. From (14) and (15) we observe that

$$|u_r(x)|^2 \le \frac{1}{2n}\sum_{i=1}^{n}\{\sum_{y_i=1}^{a_i}|\Delta_i u_r^2(x_1,\ldots,y_i,\ldots,x_n)|\}$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\{\sum_{y_i=1}^{a_i}|u_r^2(x_1,\ldots,y_i+1,\ldots,x_n) - u_r^2(x_1,\ldots,y_i,\ldots,x_n)|\}$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\{\sum_{y_i=1}^{a_i}|\Delta_i u_r(x_1,\ldots,y_i,\ldots,x_n)$$

$$\cdot\{u_r(x_1,\ldots,y_i+1,\ldots,x_n) + u_r(x_1,\ldots,y_i,\ldots,x_n)\}|\}$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\{\sum_{y_i=1}^{a_i}|\Delta_i u_r(x_1,\ldots;y_i,\ldots,x_n)$$

$$\cdot\{\Delta_i u_r(x_1,\ldots,y_i,\ldots,x_n) + 2u_r(x_1,\ldots,y_i,\ldots,x_n)\}|\}$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\{\sum_{y_i=1}^{a_i}|\{\Delta_i u_r(x_1,\ldots,y_i,\ldots,x_n)\}^2$$

$$+ 2u_r(x_1,\ldots,y_i,\ldots,x_n)\,\Delta_i u_r(x_1,\ldots,y_i,\ldots,x_n)|\}. \tag{16}$$

From (16) and on using inequality (9) repeatedly, Hölder's inequality with indices p, $\frac{p}{p-1}$ and the definition of α, we obtain

$$\Big\{\sum_{r=1}^{M} | u_r(x) |^2 \Big\}^{\frac{p}{p-1}} \tag{17}$$

$$\leq \Big(\frac{1}{2n}\Big)^{\frac{p}{p-1}} (Mn)^{\frac{p}{p-1}-1} \sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \Big\{ \sum_{y_i=1}^{a_i} | \{\Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n)\}^2$$

$$+ 2u_r(x_1, \ldots, y_i, \ldots, x_n)\, \Delta_i\, u_r(x_1, \ldots, y_i, \ldots, x_n) | \Big\}^{\frac{p}{p-1}} \Big\}$$

$$\leq \Big(\frac{1}{2n}\Big)^{\frac{p}{p-1}} (Mn)^{\frac{p}{p-1}-1} \alpha^{\frac{1}{p-1}}$$

$$\sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \Big\{ \sum_{y_i=1}^{a_i} | \{\Delta_i u_r(x_1, \ldots, y_i, \ldots, x_n)\}^2$$

$$+ 2u_r(x_1, \ldots, y_i, \ldots, x_n)\, \Delta_i\, u_r(x_1, \ldots, y_i, \ldots, x_n) |^{\frac{p}{p-1}} \Big\}\Big\}.$$

Setting $x_i = y_i$ $(i = 1, \cdots, n)$ in (17) and taking the sum over both sides of (17) with respect to $y_1, \ldots, y_n$ on B and using the definition of α, we have

$$\sum_{B} \Big\{ \sum_{r=1}^{M} | u_r(y) |^2 \Big\}^{\frac{p}{p-1}} \tag{18}$$

$$\leq \Big(\frac{1}{2n}\Big)^{\frac{p}{p-1}} (Mn\alpha)^{\frac{1}{p-1}} \alpha \sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \Big\{ \sum_{B} | \{\Delta_i u_r(y)\}^2 + 2u_r(y)\, \Delta_i\, u_r(y) |^{\frac{p}{p-1}} \Big\}\Big\}.$$

From (18) and on using the inequality (9) repeatedly, Hölder's inequality with indices $\frac{2m(p-1)}{p}$, $\frac{2m(p-1)}{2m(p-1)-p}$ and the definition of α, we have

$$\Big[\sum_{B} \Big\{ \sum_{r=1}^{M} | u_r(y) |^2 \Big\}^{\frac{p}{p-1}}\Big]^{\frac{2m(p-1)}{p}}$$

$$\leq \Big\{\Big(\frac{1}{2n}\Big)^{\frac{p}{p-1}} (Mn\alpha)^{\frac{1}{p-1}} \alpha\Big\}^{\frac{2m(p-1)}{p}} (Mn)^{\frac{2m(p-1)}{p}-1}$$

$$\sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \Big\{ \sum_{B} | \{\Delta_i u_r(y)\}^2 + 2u_r(y)\, \Delta_i\, u_r(y) |^{\frac{p}{p-1}} \Big\}^{\frac{2m(p-1)}{p}} \Big\}$$

$$\le \Big\{ (\frac{1}{2n})^{\frac{p}{p-1}} (Mn\alpha)^{\frac{1}{p-1}} \alpha \Big\}^{\frac{2m(p-1)}{p}} (Mn)^{\frac{2m(p-1)}{p} - 1}$$

$$\cdot (\alpha^n)^{\frac{2m(p-1)-p}{p}} \sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \{ \sum_{B} | \{\Delta_i u_r(y)\}^2 + 2u_r(y)\, \Delta_i\, u_r(y) |^{2m} \} \Big\}$$

$$\le (\frac{1}{2n})^{2m} (Mn\alpha)^{\frac{2m}{p}} \alpha^{\frac{2m(p-1)}{p}} (Mn)^{\frac{2m(p-1)-p}{p}}$$

$$\cdot (\alpha^n)^{\frac{2m(p-1)-p}{p}} \sum_{r=1}^{M} \Big\{ \sum_{i=1}^{n} \{ \sum_{B} 2^{2m-1} [\, | \Delta_i u_r(y) |^{4m}$$

$$+ 2^{2m} | u_r(y) |^{2m} | \Delta_i u_r(y) |^{2m} \,] \} \Big\}$$

$$= K_2 \sum_{r=1}^{M} \sum_{B} \sum_{i=1}^{n} | \Delta_i u_r(y) |^{4m} + k_3 \sum_{r=1}^{M} \sum_{B} \sum_{i=1}^{n} | u_r(y) |^{2m} | \Delta_i u_r(y) |^{2m} . \qquad (19)$$

This is the required inequality in (4) and the proof of Theorem 2 is complete.

Remark 3. We note that the hypotheses used on finite family of functions u_r ($r = 1, \cdots, M$) and the constant p in our Theorems 1 and 2 are different from those given by the authors in [3,4]. We also note that the constants k_1, k_2, k_3 involved in (2) and (4) depends on n, M, p and the size of the domain of definitions of the finite family of functions u_r, while the constant involved in (1) depends on n, p and independent of M and u_r.

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