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A COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in the unit disc $E = \{z : |z| < 1\}$. We wish to maximize $|a_3 - ua_2^2|$ over certain classes of analytic functions defined by convex subordination. This paper is concerned with the solution of the above extremal problem over certain classes of univalent analytic functions.

1. Introduction

Let U denote the class of functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

which are analytic in $E = \{z : |z| < 1\}$ and satisfying there the conditions $w(0) = 0$ and $|w(z)| < 1$.

Let S denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.2}$$

analytic and univalent in E.

Let K and S^* be the sub-classes of S which are, respectively convex and starlike in E. We shall call the function $f(z)$ of the class S an alpha-convex (α-convex) function if

(i) $\dfrac{f(z)f'(z)}{z} \neq 0$ and for any real α,

(ii) $\mathrm{Re}[(1-\alpha)\dfrac{zf'(z)}{f(z)} + \alpha(1 + z\dfrac{f''(z)}{f'})] > 0,\ z \in E.$

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Mocanu [15] introduced the concept of α-convex functions. Miller, Mocanu and Reade [14] have shown that α-convex functions are starlike in E, and for $\alpha \geq 1$, all α-convex functions are convex in E. Therefore α-convex functions are also called α-starlike functions. Concept of α-convex functions gives a continuous parametrization between starlike functions and convex function.

Al-Amiri and Read [1] introduced the class $H(\alpha)$ of analytic functions $f(z)$ in E which satisfies the condition

$$\text{Re}[(1-\alpha)f'(z) + \alpha(1 + z\frac{f''(z)}{f'(z)})] > 0.$$

Let $B(\alpha, \beta, p, g) \equiv B$ denote the class of functions $f(z)$ regular in E and defined for β real and $\alpha > 0$ by

$$f(z) = [(\alpha + i\beta)\int_0^z p(t)t^{\beta-1}g^{\alpha}(t)\,dt]^{\frac{1}{(\alpha+i\beta)}} \tag{1.3}$$

where $p(z)$ is regular, $p(0) = 1$ and $\text{Re}\,p(z) > 0$, and $g \in S^*$. (The powers appearing in (1.3) are principal values).

Bazilevic [2] showed that functions of the class B are univalent in E.

Let $M(\alpha; A, B)$ be the class of functions $f(z)$ analytic in E and satisfying the conditions

$$\frac{f(z)f'(z)}{z} \neq 0 \quad \text{and for} \quad \alpha \geq 0,$$

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + z\frac{f''(z)}{f'(z)}) < \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1. \tag{1.4}$$

To avoid repetition, we lay down, once for all that $-1 \leq B < A \leq 1$, $\alpha \geq 0$ and $z \in E$.

Let $H(\alpha; A, B)$ denote the class of functions $f(z)$ analytic in E and satisfying the condition

$$(1-\alpha)f'(z) + \alpha(1 + z\frac{f''(z)}{f'(z)}) < \frac{1+Az}{1+Bz}. \tag{1.5}$$

If we take $\beta = 0$ and $g(z) \equiv z$ in (1.3),

$$B_1(\alpha) = B(\alpha, 0, p, z) \quad \text{is the class of functions}$$

$$f(z) = [\alpha\int_0^z p(t)t^{\alpha-1}dt]^{\frac{1}{\alpha}}.$$

The class $B_1(\alpha)$ was defined by Singh in [16], and studied by Thomas in [18] and El-Ashwah and Thomas in [4,5].

Let $B_1(\alpha; A, B)$ be a sub-class of Bazilevic functions such that

$$\frac{zf'(z)(f(z))^{\alpha-1}}{z^{\alpha}} < \frac{1+Az}{1+Bz}. \tag{1.6}$$

Let $F(\alpha; A, B)$ and $G(\alpha; A, B)$ denote the classes of functions $f(z)$ analytic in E and satisfying respectively, the conditions

$$f'(z) + \alpha z f''(z) \prec \frac{1 + Az}{1 + Bz} \tag{1.7}$$

and

$$(1 - \alpha)\frac{f(z)}{z} + \alpha f'(z) \prec \frac{1 + Az}{1 + Bz}. \tag{1.8}$$

$f \in F(\alpha; A, B)$ if and only if $zf'(z) \in G(\alpha; A, B)$.

Fekete and Szegö [6] made an early study for the estimates of $|a_3 - \mu a_2^2|$ when $f(z)$ is analytic and univalent in E. The well-known result due to them states that if $f(z)$ is analytic univalent in E, then

$$| a_3 - \mu a_2^2 | \leq \begin{cases} 4\mu - 3, & \mu \geq 1, \\ 1 + 2\exp(\dfrac{-2\mu}{1 - \mu}), & 0 \leq \mu \leq 1, \\ 3 - 4\mu, & \mu \leq 0. \end{cases}$$

Hummel [10,11] proved the conjecture of V.Singh that $|a_3 - a_2^2| \leq \frac{1}{3}$ for the class K of convex functions. Keogh and Merkes [13] obtained sharp estimates for $|a_3 - \mu a_2^2|$ when f is close-to-convex, starlike and convex in E. For μ complex Szynal [17] obtained the estimates for $|a_3 - \mu a_2^2|$ for the Mocanu class $M(\alpha)$. Al-Amiri and Reade [1] obtained estimates for $|a_3 - \mu a_2^2|$ for the class $H(\alpha)$ when μ is complex. Singh [16] obtained the estimates for $|a_3 - \mu a_2^2|$ for the class $B_1(\alpha)$.

The following observations are obvious.

(i) $M(\alpha; 1, -1) \equiv M(\alpha)$, class of α-convex functions,

(ii) $M(0; A, B) \equiv B_1(0; A, B) \equiv S^*(A, B)$, a sub-class of starlike functions studied by Janowski [12], and the authors [7] obtained some coefficient estimates for the class $S^*(A, B)$;

(iii) $M(1; A, B) \equiv K(A, B)$, a sub-class of convex functions studied by the authors in [8], (However, estimates of $|a_3 - \mu a_2^2|$ for the classes $S^*(A, B)$ and $K(A, B)$ remained unproved)

(iv) $M(0; 1, -1) \equiv B_1(0; 1, -1) \equiv S^*$,

(v) $M(1; 1, -1) \equiv H(1; 1, -1) \equiv K$,

(vi) $H(\alpha; 1, -1) \equiv H(\alpha)$,

(vii) $H(0; A, B) \equiv B_1(1; A, B) \equiv R(A, B)$, a sub-class of univalent analytic functions studied by the authors in [9],

(viii) $F(\alpha; 1, -1) \equiv F(\alpha)$ and $G(\alpha; 1, -1) \equiv G(\alpha)$, classes considered by Chichra in [3];

(ix) $H(0; 1, -1) \equiv B_1(1; 1, -1) \equiv F(0; 1, -1) \equiv G(1; 1, -1) \equiv R$. R is the Noshiro-Warschawski class studied by several authors.

In this paper, we obtain sharp estimates for $|a_3 - \mu a_2^2|$ when $f \in M(\alpha; A, B)$ or $H(\alpha; A, B)$ or $F(\alpha; A, B)$ or $G(\alpha; A, B)$ or $B_1(\alpha; A, B)$.

Results due to Keogh and Merkes [13], Szynal [17], Al-Amiri and Reads [1], and Singh [16] follow as special cases from our theorems.

2. Coefficient inequality

Theorem 2.1. *If $f \in M(\alpha; A, B)$, then*
(i) *for μ complex*

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \dfrac{(A - B)}{2(1 + 2\alpha)}, & | \mu - \gamma | \leq v, \quad (2.1) \\ \dfrac{(A - B)^2}{(1 + \alpha)^2} | \mu - \gamma |, & | \mu - \gamma | \geq v; \quad (2.2) \end{cases}$$

and
(ii) for μ real,

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \dfrac{(A - B)^2}{(1 + \alpha)^2}(\gamma - \mu), & \mu \leq \gamma - v, \quad (2.3) \\ \dfrac{(A - B)}{2(1 + 2\alpha)}, & \gamma - v \leq \mu \leq \gamma + v, \quad (2.4) \\ \dfrac{(A - B)^2}{(1 + \alpha)^2}(\mu - \gamma), & \mu \geq \gamma + v; \quad (2.5) \end{cases}$$

where

$$\gamma = \frac{(A - B)(1 + 3\alpha) - B(1 + \alpha)^2}{2(1 + 2\alpha)(A - B)}. \tag{2.6}$$

$$v = \frac{(1 + \alpha)^2}{2(1 + 2\alpha)(A - B)}. \tag{2.7}$$

All the estimates are sharp.

Proof. From (1.4), by definition of subordination,

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha(1 + z\frac{f''(z)}{f'(z)}) = \frac{1 + Aw(z)}{1 + Bw(z)}. \tag{2.8}$$

By expanding (2.8) and equating the coefficients we have

$$a_2 = \frac{(A - B)c_1}{1 + \alpha} \tag{2.9}$$

and

$$a_3 = \frac{(A - B)}{2(1 + 2\alpha)}c_2 + [\frac{(1 + 3\alpha)a_2^2 - B(A - B)c_1^2}{2(1 + 2\alpha)}]. \tag{2.10}$$

From (2.9) and (2.10), we get

$$a_3 - \mu a_2^2 = \frac{(A-B)}{2(1+2\alpha)}c_2 + \frac{(A-B)^2}{(1+\alpha)^2}\left[\frac{(A-B)(1+3\alpha)-B(1+\alpha)^2}{2(1+2\alpha)(A-B)} - \mu\right]c_1^2.$$

Therefore

$$| a_3 - \mu a_2^2 | \leq \frac{(A - B)}{2(1 + 2\alpha)} | c_2 | + \frac{(A - B)^2}{(1 + \alpha)^2} | \gamma - \mu || c_1 |^2 . \tag{2.11}$$

Also

$$| c_2 | \leq 1 - | c_1 |^2, \tag{2.12}$$

$$| c_1 | \leq 1. \tag{2.13}$$

(2.11) and (2.12) lead us to

$$| a_3 - \mu a_2^2 | \leq \frac{(A - B)}{2(1 + \alpha)} + \frac{(A - B)^2}{(1 + \alpha)^2} [| \gamma - \mu | - v] | c_1 |^2 . \tag{2.14}$$

If $|\gamma - \mu| \leq v = \dfrac{(1+\alpha)^2}{2(1+2\alpha)(A-B)}$, (2.1) follows.
If $|\gamma - \mu| \geq v = \dfrac{(1+\alpha)^2}{2(1+2\alpha)(A-B)}$, (2.2) follows, since $|c_1| \leq 1$.
Consider the case when μ is real.

Case I. $\mu \leq \gamma$.
From (2.14), we have

$$| a_3 - \mu a_2^2 | \leq \frac{(A - B)}{2(1 + 2\alpha)} + \frac{(A - B)^2}{(1 + \alpha)^2} [(\gamma - v) - \mu] | c_1 |^2 . \tag{2.15}$$

If $\mu \leq (\gamma - v)$, from (2.15) we have

$$\begin{aligned} | a_3 - \mu a_2^2 | &\leq \frac{(A - B)}{2(1 + 2\alpha)} + \frac{(A - B)^2}{(1 + \alpha)^2}(\gamma - \mu - v), \text{ since } | c_1 | \leq 1 \\ &= \frac{(A - B)^2}{(1 + \alpha)^2}(\gamma - \mu). \end{aligned}$$

If $\gamma - v \leq \mu \leq \gamma$, we obtain (2.4) from (2.15).

Case II. $\mu \geq \gamma$.
From (2.14) we have

$$| a_3 - \mu a_2^2 | \leq \frac{(A - B)}{2(1 + 2\alpha)} + \frac{(A - B)^2}{(1 + \alpha)^2} [\mu - \gamma - v] | c_1 |^2, \ | c_1 | \leq 1. \tag{2.16}$$

If $\mu \leq \gamma + v$, (2.4) follows. If $\mu \geq \gamma + v$, we obtain (2.5) from (2.16). The estimates (2.1) and (2.4) are sharp for the function $f_{0(1)}(z)$ defined by

$$f_{0(1)}(z) = \begin{cases} [\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}(1+B\delta t^2)^{\frac{(A-B)}{2\alpha B}}dt]^\alpha, & B \neq 0, \\ [\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}\exp(\frac{A\delta t^2}{2\alpha})dt]^\alpha, & B = 0. \end{cases} \quad |\delta| = 1$$

The estimates (2.2), (2.3), (2.5) are sharp for the function $f_{1(1)}(z)$ defined by

$$f_{1(1)}(z) = \begin{cases} [\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}(1+\delta Bt)^{\frac{(A-B)}{\alpha B}}dt]^\alpha, & B \neq 0, \\ [\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}\exp(\frac{\delta At}{\alpha})dt]^\alpha, & B = 0. \end{cases} \quad |\delta| = 1$$

On the same lines we have

Theorem 2.2. *Let $f \in H(\alpha; A, B)$, then*
(i) *for μ complex,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)}{3(1+\alpha)}, & |\gamma_1 - \mu| \leq \frac{4}{3(1+\alpha)(A-B)}, \quad (2.17) \\ \frac{(A-B)^2}{4}|\gamma_1 - \mu|, & |\gamma_1 - \mu| \geq \frac{4}{3(1+\alpha)(A-B)}; \quad (2.18) \end{cases}$$

and
(ii) *for μ real,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(A-B)^2}{4}(\gamma_1 - \mu), & \mu \leq \frac{4[\alpha(A-B)-(1+B)]}{3(1+\alpha)(A-B)}, \quad (2.19) \\ \frac{(A-B)}{3(1+\alpha)}, & \frac{4[\alpha(A-B)-(1+B)]}{3(1+\alpha)(A-B)} \leq \mu \leq \frac{4[\alpha(A-B)+(1-B)]}{3(1+\alpha)(A-B)}, \quad (2.20) \\ \frac{(A-B)^2}{4}(\mu - \gamma_1), & \mu \geq \frac{4[\alpha(A-B)+(1-B)]}{3(1+\alpha)(A-B)}; \quad (2.21) \end{cases}$$

where

$$\gamma_1 = \frac{4[\alpha A - (1+\alpha)B]}{3(1+\alpha)(A-B)}.$$

From (1.5), we have

$$(1-\alpha)f'(z) + \alpha(1 + z\frac{f''(z)}{f'(z)}) = \frac{1+Aw(z)}{1+Bw(z)}.$$

Bounds (2.17) *and* (2.20) *are sharp for* $w(z) = z^2$. (2.18), (2.19) *and* (2.21) *are sharp for* $w(z) = z$.

Theorem 2.3. *Let* $f \in B_1(\alpha; A, B)$, *then*
(i) *for any complex number* μ,

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \dfrac{(A-B)}{2+\alpha}, | \gamma_2 - \mu | & \leq \frac{(1+\alpha)^2}{(2+\alpha)(A-B)}, \quad (2.22) \\ \dfrac{(A-B)^2}{(1+\alpha)^2} | \gamma_2 - \mu |, & | \gamma_2 - \mu | \geq \frac{(1+\alpha)^2}{(2+\alpha)(A-B)}; \quad (2.23) \end{cases}$$

and
(ii) *for any real number* μ,

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \dfrac{(A-B)^2}{(1+\alpha)^2}(\gamma_2 - \mu), & \mu \leq \frac{(1-\alpha)(2+\alpha)(A-B)-2(1+B)(1+\alpha)^2}{2(2+\alpha)(A-B)}, \quad (2.24) \\ \dfrac{(A-B)}{2+\alpha}, & \frac{(1-\alpha)(2+\alpha)(A-B)-2(1+B)(1+\alpha)^2}{2(2+\alpha)(A-B)} \\ & \leq \mu \leq \frac{(1-\alpha)(2+\alpha)(A-B)+2(1-B)(1+\alpha)^2}{2(2+\alpha)(A-B)}, \quad (2.25) \\ \dfrac{(A-B)^2}{(1+\alpha)^2}(\mu - \gamma_2), & \mu \geq \frac{(1-\alpha)(2+\alpha)(A-B)+2(1-B)(1+\alpha)^2}{2(2+\alpha)(A-B)}; \quad (2.26) \end{cases}$$

where

$$\gamma_2 = \frac{(1-\alpha)(2+\alpha)(A-B) - 2B(1+\alpha)^2}{2(2+\alpha)(A-B)}.$$

Equality signs in (2.22) *and* (2.25) *hold for the function*

$$f_{0(2)}(z) = [\alpha \int_0^z t^{\alpha-1}(\frac{1 + A\delta t^2}{1 + B\delta t^2})dt]^{\frac{1}{\alpha}}, \qquad | \delta | = 1.$$

Equality signs in (2.23), (2.24) *and* (2.26) *are attained by the function*

$$f_{1(2)}(z) = [\alpha \int_0^z t^{\alpha-1}(\frac{1 + A\delta t}{1 + B\delta t})dt]^{\frac{1}{\alpha}}, \qquad | \delta | = 1.$$

Theorem 2.4. *If* $f \in F(\alpha; A, B)$, *then*
(i) *for* μ *complex*,

$$| a_3 - \mu a_2^2 | \leq \begin{cases} \dfrac{(A-B)}{3(1+2\alpha)}, & | \mu + \gamma_3 | \leq \frac{4(1+\alpha)^2}{3(A-B)(1+2\alpha)}, \quad (2.27) \\ \dfrac{(A-B)^2}{4(1+\alpha)^2} | \mu + \gamma_3 |, & | \mu + \gamma_3 | \geq \frac{4(1+\alpha)^2}{3(A-B)(1+2\alpha)}; \quad (2.28) \end{cases}$$

and

(ii) *for μ real*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{-(A-B)^2}{4(1+\alpha)^2}(\mu+\gamma_3), & \mu \leq \frac{-4(1+B)(1+\alpha)^2}{3(A-B)(1+2\alpha)}, \quad (2.29) \\ \dfrac{(A-B)}{3(1+2\alpha)}, & \frac{-4(1+B)(1+\alpha)^2}{3(A-B)(1+2\alpha)} \leq \mu \leq \frac{4(1-B)(1+\alpha)^2}{3(A-B)(1+2\alpha)}, \quad (2.30) \\ \dfrac{(A-B)^2}{4(1+\alpha)^2}(\mu+\gamma_3), & \mu \geq \frac{4(1-B)(1+\alpha)^2}{3(A-B)(1+2\alpha)}; \quad (2.31) \end{cases}$$

where

$$\gamma_3 = \frac{4(1+\alpha)^2 B}{3(1+2\alpha)(A-B)}. \tag{2.32}$$

All the bounds are sharp.

Proof. From (1.7), we have

$$f'(z) + \alpha z f''(z) = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Equating the coefficients of z and z^2,

$$a_2 = \frac{(A-B)}{2(1+\alpha)} c_1$$

and

$$a_3 = \frac{(A-B)}{3(1+2\alpha)}(c_2 - Bc_1^2).$$

So that

$$a_3 - \mu a_2^2 = \frac{(A-B)}{3(1+2\alpha)} c_2 - \frac{(A-B)^2}{4(1+\alpha)^2}\left[\mu + \frac{4B(1+\alpha)^2}{3(1+2\alpha)(A-B)}\right] c_1^2.$$

Therefore

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3(1+2\alpha)} |c_2| + \frac{(A-B)^2}{4(1+\alpha)^2} \left|\mu + \frac{4B(1+\alpha)^2}{3(1+2\alpha)(A-B)}\right| |c_1|^2$$

which together with (2.12) gives

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{3(1+2\alpha)} + \frac{(A-B)^2}{4(1+\alpha)^2}\left[|\mu + \gamma_3| - \frac{4(1+\alpha)^2}{3(1+2\alpha)(A-B)}\right] |c_1|^2 \tag{2.33}$$

If $|\mu + \gamma_3| \leq \dfrac{4(1+\alpha)^2}{3(1+2\alpha)(A-B)}$, (2.27) at once follows.

If $|\mu+\gamma_3| \geq \dfrac{4(1+\alpha)^2}{3(1+2\alpha)(A-B)}$, we get (2.30).
Consider the case when μ is real.

Case I. $\mu+\gamma_3 \geq 0$.
(2.33) reduces to

$$\begin{aligned} | a_3 - \mu a_2^2 | &\leq \frac{(A-B)}{3(1+2\alpha)} + \frac{(A-B)^2}{4(1+\alpha)^2}[\mu+\gamma_3 - \frac{4(1+\alpha)^2}{3(1+2\alpha)(A-B)}] \mid c_1 \mid^2 \\ &= \frac{(A-B)}{3(1+2\alpha)} + \frac{(A-B)^2}{4(1+\alpha)^2}[\mu - \frac{4(1-B)(1+\alpha)^2}{3(1+2\alpha)(A-B)}] \mid c_1 \mid^2, \mid c_1 \mid \leq 1. \end{aligned}$$

If $\mu \geq \dfrac{4(1-B)(1+\alpha)^2}{3(1+2\alpha)(A-B)}$, (2.31) follows.
If $\mu \leq \dfrac{4(1-B)(1+\alpha)^2}{3(1+2\alpha)(A-B)}$, we have

$$| a_3 - \mu a_2^2 | \leq \frac{(A-B)}{3(1+2\alpha)}. \tag{2.34}$$

Case II. $\mu+\gamma_3 \leq 0$.
(2.33) takes the form

$$| a_3 - \mu a_2^2 | \leq \frac{(A-B)}{3(1+2\alpha)} - \frac{(A-B)^2}{4(1+\alpha)^2}[\mu + \frac{4(1+\alpha)^2(1+B)}{3(1+2\alpha)(A-B)}] \mid c_1 \mid^2, \mid c_1 \mid \leq 1. \tag{2.35}$$

If $\mu \leq \dfrac{-4(1+\alpha)^2(1+B)}{3(1+2\alpha)(A-B)}$, we get (2.29).
Again

$$| a_3 - \mu a_2^2 | \leq \frac{(A-B)}{3(1+2\alpha)} \quad \text{if} \quad \frac{-4(1+\alpha)^2(1+B)}{3(1+2\alpha)(A-B)} \leq \mu. \tag{2.36}$$

(2.34) and (2.36) together yield (2.30).
Extremal function for (2.27) and (2.30) is given by

$$\begin{aligned} f_{0(3)}(z) &= P_0(z) * k_0(z) \text{ where} \\ P_0(z) &= \int_0^z \frac{1+At^2}{1+Bt^2} dt \end{aligned}$$

and

$$\begin{aligned} k_0(z) &= \frac{1}{\alpha} z^{(\frac{1}{\alpha})-1} \int_0^z \frac{t^{(\frac{1}{\alpha})-1}}{1-t} dt \\ &= z + \sum_{k=2}^{\infty} \frac{1}{1+(k-1)\alpha} z^k. \end{aligned} \tag{2.37}$$

Extremal functions for the bounds (2.28), (2.29) and (2.31) are given by

$$f_{1(3)}(z) = P_1(z) * k_0(z) \text{ where}$$
$$P_1(z) = \int_0^z \frac{1 + At}{1 + Bt} dt.$$

* denotes the convolution or Hadamard product of two functions.

The following theorem is an easy consequence of Theorem 2.4.

Theorem 2.5. *If $f \in G(\alpha; A, B)$, then*

(i) *for any complex number μ,*

$$| a_3 - \mu a_2^2 | \le \begin{cases} \dfrac{(A - B)}{(1 + 2\alpha)}, & | \mu + \gamma_4 | \le \dfrac{(1+\alpha)^2}{(A-B)(1+2\alpha)}, \quad (2.38) \\ \dfrac{(A - B)^2}{(1 + \alpha)^2} | \mu + \gamma_4 |, & | \mu + \gamma_4 | \ge \dfrac{(1+\alpha)^2}{(A-B)(1+2\alpha)}, \quad (2.39) \end{cases}$$

and

(ii) *for any real number μ,*

$$| a_3 - \mu a_2^2 | \le \begin{cases} \dfrac{-(A - B)^2(\mu + \gamma_4)}{(1 + \alpha)^2}, & \mu \le \dfrac{-(1+B)(1+\alpha)^2}{(A-B)(1+2\alpha)}, \quad (2.40) \\ \dfrac{(A - B)}{(1 + 2\alpha)}, & \dfrac{-(1+B)(1+\alpha)^2}{(1+2\alpha)(A-B)} \le \mu \le \dfrac{(1-B)(1+\alpha)^2}{(1+2\alpha)(A-B)}, \quad (2.41) \\ \dfrac{(A - B)^2}{(1 + \alpha)^2}(\mu + \gamma_4), & \mu \ge \dfrac{(1-B)(1+\alpha)^2}{(A-B)(1+2\alpha)}; \quad (2.42) \end{cases}$$

where

$$\gamma_4 = \frac{B(1 + \alpha)^2}{(A - B)(1 + 2\alpha)}.$$

Extremal function $f_{0(4)}(z)$ for the bounds (2.38) *and* (2.41) *is defined by*

$$f_{0(4)}(z) = z(\frac{1 + Az^2}{1 + Bz^2}) * k_0(z).$$

Extremal function $f_{1(4)}(z)$ for the bounds (2.39), (2.40) and (2.42) is defined by

$$f_{1(4)}(z) = z(\frac{1 + Az}{1 + Bz}) * k_0(z).$$

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