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SOME REMARKS ON THE FINITENESS CONDITIONS OF RINGS

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Abstract. The aim of this paper is to study the finiteness of rings. We prove that if A is a regular left-self-injective ring, then A is of type III (purely infinite) implies that $E(A[x])$ is, and A contains an abelian idempotent if and only if $E(A[x])$ contains an abelian idempotent. Also we prove that.

If A is a regular left self-injective ring and J is a left ideal in $A[x]$ such that $C(J)$ is an essential left ideal in A, then there exists a countably generated left ideal J' in $A[x]$ such that $C(J')$ is an essential left ideal in A, and if J' is an essential left ideal in $A[x]$, then J is an essential left ideal in $A[x]$.

1. Introduction

As special case of the work in [6], we have that "A regular left self-injective ring A is of type I, I_f, I_∞ if and only if $E(A[x])$ is of the same respective type. In this paper we are interested in study the relation between the finiteness of the regular left self-injective rings and the finiteness of the injective envelope of their polynomial rings.

Throughout this paper all rings are associative with unit. A ring A is regular provided that for every $x \in A$, there exists $y \in A$ such that $xyx = x$. A ring A is unit-regular provided that, for each $x \in A$, there is an invertable element $u \in A$ such that $x = xux$. A regular ring is abelian provided all idempotents in A are central. A ring A is called left self-injective if ${}_AA$ is injective A-module. A module M is directly finite if M is not isomorphic to any proper direct summand of itself and a ring A is directly finite if ${}_AA$ is directly finite A-module and it is directly infinite if it is not directly finite. An idempotent e in a ring A is called faithful in A if 0 is the only central idempotent of A which is orthogonal to e. An idempotent, e in a regular ring A is called abelian (directly finite) idempotent if $e\,A\,e$ is abelian regular (directly finite) ring. A regular, left self-injective ring is said to be purely infinite if it contains no nonzero directly finite central idempotent. And it is said to be of type II if it contains a faithful directly finite idempotent but it contains no nonzero abelian idempotents. Moreover it is called of type II_f (II_∞) if it is of type II and directly finite (purely infinite). And it is called of type III if it contains no nonzero directly finite idempotents.

A ring A is said to be a Baer ring if every left (right) annihilator is of the form Ae (eA), e is an idempotent in A. The central·cover of an element $x \in A$, written $C(x)$, is the smallest central idempotent e in A satisfying $e\,x = x$. By the injective invelope of a ring A we mean the injective envelope of it as a left module over itself and we denoted it by $E(A)$. Finally the set of all central idempotents in a ring A is denoted by $B(A)$.

2. Preliminaries

In this section we collect some results and a consequences of them, will be needed in this paper.

(2.1) Let A be a Baer ring, e be an idempotent in A and f be an idempotent in $e\,A\,e$. If f is a directly finite idempotent in $e\,A\,e$, then f is a directly finite idempotent in A.

(2.2) Let e, f be idempotents in a Baer ring A. Then the following conditions are equivalent

(i) $A\,e$ is isomorphic to $A\,f$ as a left A-modules

(ii) $e\,A$ is isomorphic to $f\,A$ as a right A-modules

(iii) There exist elements x and y such that $x\ e A f$, $y\ f A e$, $x\,y = e$ and $y\,x = f$

Two idempotent e and f in a Baer ring A are called equivalent and denoted by $e \sim f$ if they satisfy the equivalence conditions (2.2).

(2.3) A Baer ring A is directly finite if and only if for every idempotent $e \in A$, $e \sim l$ implies $e = l$.

(2.4) Let A be a Baer ring with no nonzero nilpotent ideals. If $e = e^2 \in A$ and $f \in B(e\,A\,e)$, then $C(f)\ e = f$

The proof of 2.1, 2.2, 2.3 and 2.4 can be found in [8]

Notice that a regular ring is a Baer ring if and only if the lattice of principal left (ring) ideals of it is complete and hence every regular left (ring) self-injective ring is a Baer ring. Therefore (2.1), (2.2), (2.3) and (2.4) are satisfied for regular left self-injective rings.

(2.5) Let A be a left nonsingular ring, N be a submodule of the left A-module M. Then N is essential in M $(N \leq_e M)$ if and only if $N\,x^{-1}$ is an essential left ideal in A for each $x \in M$, where $N\,x^{-1} = \{a \in A : a\,x \in N\}$.

The proof of (2.5) can be in [5].

A consequence of (2.5) is that if I is essential left ideal of a left nonsingular ring A, then for each $b \in A$, there exists an essential left ideal K of A such that $0 \neq K\ b \subset I$.

(2.6) Let A be a regular, left self-injective ring, and let $\{J_i\}$ be an independent family of left ideals of A. Then there exist orthogonal idempotents $e_i \in A$ such that each $J_i \leq_{.e} A\,e_i$. If the J_i are also principal, then each $J_i = A\,e_i$.

The proof of (2.6) can be found in [2].

Let A be a regular left self-injective ring and I be a left ideal in A. If $\{K_i\}$ is a maximal independent family of principal left ideals of A such that $K_i \subset I$ for each i, then $\oplus K_i \leq_e I$. By using (2.6) we can find an orthogonal idempotents $\{e_i\}$ such that $Ae_i = K_i$ for each i. Therefore $\oplus Ae_i \leq_e I$. On the other hand $E({}_AI)$ is a direct summand in A which implies the existance of an idempotent $e \in A$ such that $I \leq_e Ae$ Moreover e is unique.

(2.7) Every regular, left self-injective ring satisfies general comparability.

The proof of (2.7) can be found in [9]

(2.8) If A is a unit-regular ring, then every finitely generated projective A-module is directly finite, consequently, $M_n(A)$ is directly finite for all n

The proof of (2.8) can be found in [4].

(2.9) If A is a directly finite regular ring satisfying general comparability, then A is unit-regular.

The proof of (2.9) can be found in [10].

From (2.7) and (2.9) a directly finite regular left self-injective ring is a unit-regular. Conversely if A is a unit-regular left self-injective, then ${}_AA$ is a finitely generated non-singular left A-module, which implies that ${}_AA$ is projective, hence (2.8) shows that A is a directly finite.

(2.10) Let A be a semiprime, left nonsingular ring and e be an idempotent of A. Then $eE(A)e$ is the injective envelope of eAe

The proof of 2.10 can be found in [1]

(2.11) Let A be a regular, left self-injective ring. If A is directly finite, then every nonzero ideal of A contains a nonzero central idempotent.

(2.12) For a regular left self-injective ring A the following conditions are equivalent.

(i) A is purely infinite

(ii) $nA^A \lesssim A^A$ for some integer $n \geq 2$

(iii) $nA^A \simeq A^A$ for all positive integer n

(iv) $E(\chi_o A^A) \simeq A^A$

The proof of (2.11) and (2.12) can be found in [9]

(2.13) Let A be a regular left self-injective ring, $\sigma \in \mathrm{Aut}(A)$ and D be a σ-derivation of A. Then $B(E(A[X, \sigma, D])) = (B(A))^\sigma$, where $(B(A))^\sigma = \{e \in B(A) : \sigma(e) = e\}$.

(2.14) If A is an abelian regular ring, then $E(A[X, D])$ is an abelian regular ring.

The proof of (2.13) and (2.14) can be found in [6].

3. Finiteness conditions of rings

Notice first that, if A is a left nonsingular ring, then $A[x]$ is left nonsingular and hence $E(A[X])$ is a regular left self-injective ring.

Proposition 3.1. *If A is a regular left self-injective ring of type III, then $E(A[X])$ is of type III.*

Proof. Assume that $E(A[X])$ is not of type III hence there exists a non zero directly finite idempotent f in $E(A[X])$, which implies that $E(A[X])f$ is directly finite and $I = I = E(A[X])f \cap A[X]$ is a non zero left ideal in $A[X]$, since A is a regular ring, then there exists a non zero polynomial p in I with minimal length and $p = e\,X^n + a_{n-1}\,X^{n-1} + \cdots + a_0$. Moreover $p = e\,p = e\,X^n + e\,a_{n-1}\,X^{n-1} + \cdots + a_0$, $e = e^2 \in A$ we define

$$\varphi : A[X] \to A[X]P$$
$$q \to q \cdot P$$

It is clear that φ is an epimorphism with $\mathrm{Ker}\,\varphi = \mathrm{Ann}\,(P) = A[X](1-e)$, which implies that $A[X]P \simeq A[X]e$, hence $E(A[X])e \simeq E(A[X]P) \subseteq E(A[X])f$, which implies that $E(A[X])e$ is directly finite, therefore $A\,e$ is directly finite and hence e is a directly finite idempotent in A which is a contradiction.

Proposition 3.2. *Let A be a regular left self-injective ring. If A is purely infinite, then $E(A[X])$ is purely infinite.*

Proof. From (2.13), we have $B(E(A[X])) = B(A)$, and hence the proof of the proposition is clear.

Recal that if A is a Baer ring and $e = e^2 \in A$ is a nonzero idempotent, then A is of type III implies that $e\,A\,e$ is of type III (see [8]). Therefore this result is also true for regular left self-injective rings.

Example 3.3. This example shows that the previous result is not true for purely infinite regular left self-injective rings. Let K be a field and $A = \mathrm{End}_K\,(K^N)$, then A is a purely infinite regular left self-injective ring. We define $P_0 : K^N \to K$, P_0 is a projection and $P_0\,A\,P_0 \simeq K$ is directly finite.

Proposition 3.4. *Let A be a regular left self-injective ring and f be a nonzero central idempotent in A. If A is purely infinite, then $f\,A\,f$ is purely infinite.*

Proof. Since A is purely infinite ring, then (2.12) implies that $A \simeq E(\oplus_i Ae_i)$ with $A \simeq Ae_i$ for each $i \in N$ and the $e_i'^s$ are idempotents in A. By using (2.2), we have that, for each $i \in N$, there exist $x_i \in Ae_i, y_i \in e_i\,A$ such that $x_i y_i = 1$ and $y_i x_i = e_i$. Which implies that $(fx_if)\,(fy_if) = f, (fy_if)(fx_if) = e_if$ with $fxf \in fAe_if = Ae_if$ and $fyf \in e_ifA$, hence $Af \simeq Ae_if$ for each $i \in N$. Since A is injective, there exists an idempotent h in A such that $E(\oplus_i Ae_i) \simeq Ah$ hence $A \simeq Ah$, which implies that $Af \simeq Ahf$ therefore $Af \simeq E(\oplus_i Ae_if)$.

Condition * We say that a ring A satisfies condition * if for each sequence $e_1, e_2, \ldots, e_n, \ldots$ of nonzero idempotents in A, there exists $k \in N$ such that $C(e_k) \in e_{k-1}\,A\,e_{k-1}$, where $C(e_k)$ is the central cover of e_k in A.

Notice that, every division ring satisfies condition *. Also there exists a regular left self-injective ring which does not satisfy condition * as the following example shows.

Example 3.5. Let K be a field and $A = \mathrm{End}_k\,(K^N)$ then A is a regular left self-injective ring.

Let $\{f_n : n \in N\}$ be a base of K^N as a vector space over K, we define $V_0 = K^N$, $V_1 = \oplus_{n \in N}(f_{2n}K)$, $V_2 = \oplus_{n \in N}(f_{2^2n}K), \ldots$, $V_p = \oplus_{n \in N}(f_{2^pn}K)$, then $V_0 \supseteq V_1 \supseteq V_2 \supseteq \ldots V_n \supseteq \ldots$, let $P_n : V_0 \to V_n$ be the projections from V_0 into V_n for each $n \in N$, then $P_1, P_2, \ldots, P_n, \ldots$ is a sequence of idempotents in A and $\{0, 1\}$ is the set of all central idempotents in A.

Proposition 3.6. *If A is a regular left self-injective ring such that $E(A[X])$ satisfies condition *, then A is purely infinite implies that $E(A[X])$ is of type III.*

Proof. Assume that $E(A[X])$ is not of type III, hence $E(A[X])$ contains a directly finite idempotent say e_1. Therefore (2.11) implies that every nonzero ideal in $e_1 E(A[X])e_1$ contains a nonzero central idempotent in $e_1 E(A[X])e_1$. Since every regular ring is a semiprime, then by using (2.4), there exists a central idempotent e_2 in $e_1 E(A[X])e_1$ such that $C(e_2)e_1 = e_2$, hence $e_2E(A[X])e_2 = e_1 C(e_2)E(A[X])C(e_2)e_1 \subseteq e_1E(A[X])e_1$. Therefore $e_2 E(A[X])e_2$ is a directly finite regular left self-injective ring, which implies the existance of an idempotent $e_3 \in e_2 E(A[X]) e_2$ which is central in $e_2 E(A[X])e_2$ and directly finite in $E(A[X])$. And so on, we obtain a sequence of idempotents $e_1, e_2, \ldots, e_n, \ldots$ in $E(A[X])$ such that, each e_i is directly finite in $E(A[X])$ and each e_i is central in $e_{i-1} E(A[X]) e_{i-1}$. Therefore there exists e_k for some $k \in N$, such that $C(e_k)$ is contained in $e_{k-1} E(A[X])e_{k-1}$ and $e_k = C(e_k)e_{k-1}$. Moreover $C(e_k)(e_{k-1}E(A[X])e_{k-1})C(e_k) = e_kE(A[X])e_k$, which implies that $C(e_k)$ is a directly finite idempotent in $e_{k-1}E(A[X])\ e_{k-1}$. Therefore (2.4) implies that $C(e_k)$ is a central directly finite idempotent in $E(A[X])$, moreover (2.13) implies that $C(e_k)$ is a central directly finite idempoten in A which is a contradiction.

Remark 3.7. The proof of proposition 3.6 shows that, if $E(A[X])$ satisfies the hypothesis of the proposition, then $E(A[X])$ contains directly finite idempotent, implies that it contains a central directly finite idempotent.

Notice that for unit-regular ring, the statement A is directly finite implies that $E(A)$ is directly finite is not always true as the following example shows.

Example 3.8. Let K be a field, then $M_{2^n}(K)$ is a regular directly finite ring for each $n \in N$. For each $n \in N$, we identify for each element X in $M_{2^n}(K)$, the element $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ in $M_{2^{n+1}}(K)$ and hence $M_{2^n}(K) \subseteq M_{2^{n+1}}(K) \subseteq \cdots$ for each $n \in N$. Since the direct limit of regular rings is a regular ring, and also the direct limit of directly finite rings is a directly finite ring, then $A = \lim_{\to} M_{2^n}(K)$ is a regular directly finite ring. But the left injective envelope of A and the right injective envelope of A are different, both of which are of type III (see [3] theorem II.3.1) Therefore $E(A)$ is not directly finite.

Proposition 3.9. *If A is a left non singular ring such that $M_n(A)$ is directly infinite for some $n \in N$, $n \geq 2$, then $E(A)$ is a directly infinite.*

Proof. Since A is left nonsingular, then the maximal left quotient ring $Q(A)$ is regular left self-injective ring. If $E(A)$ is directly finite, then $Q(A)$ is also directly finite and hence unit-regular. Therefore (2.8) implies that $M_n(Q(A))$ is directly finite for each $n \in N, n \geq 2$, whence $M_n(A)$ is directly finite for each $n \in N$, $n \geq 2$, which is a contradiction.

If $A = \lim M_{2^n}(E)$ as in example 3.8, then $A[X]$ is directly finite, but $E(A[X])$ is not directly finite. We discuss now that, if A is directly finite regular left self-injective ring, is $E(A[X])$ is directly finite.

Theorem 3.10. *If A is a nonsingular ring, then $E(A[X])$ is directly infinite if and only if there exists left ideals I and J in $A[X]$ such that $I \simeq J$, $I \leq_e A[X]$ and $J \not\leq_e A[X]$ as left $A[X]$-modules.*

Proof. Notice first that $E(A[X])$ is a Baer ring
(1) Assume that, there exist left ideals I, J in $A[X]$ such that $I \simeq J$, $I \leq_e A[X]$ and $J \not\leq_e A[X]$ as left $A[X]$-modules. Hence $E(I)$ and $E(J)$ are direct summands of $E(A[X])$, moreover $I \leq_e A[X]$ implies that $E(I) = E(A[X])$, also $E(J) = E(A[x])f$ for some nonzero idempotent f in $E(A[X])$ with $f \neq 1$. But $I \simeq J$ implies that $E(I) \simeq E(J)$, hence $f \sim 1$. Therefore (2.3) shows that $E(A[X])$ is a directly infinite ring.
(2) Assume that $E(A[X]) = B$ is directly infinite then there exists idempotent f in B such that $f \sim 1$ and $0 \neq f \neq 1$. Therefore there exists an isomorphism $\psi : Bf \to B$ of left B-modules. We have that $Bf \cap A[X] \leq_e Bf$ as left $A[X]$-modules, since ψ is an isomorphism, $\psi(Bf \cap A[X]) \leq_e B$, which implies that

$$\psi(Bf \cap A[x]) \cap A[X] \leq_e A[X] \qquad (*)$$

Also we have that $\psi(Bf \cap A[X]) \cap A[x] \leq_e B$. Since ψ^{-1} is an isomorphism, then

$$J = \psi^{-1}(\psi(Bf \cap A[X] \cap A[X]) \leq_e Bf.$$

Now

$$\begin{aligned} J &\subseteq \psi^{-1}(\psi(Bf \cap A[x])) \cap \psi^{-1}(A[x]) \\ &\subseteq (Bf \cap A[X]) \cap Bf \\ &\subseteq A[X] \end{aligned}$$

also $J \cap (B(1-f) \cap A[X]) \subseteq (Bf \cap A[X]) \cap (B(1-f) \cap A[X]) \subseteq Bf \cap B(1-f) = 0$, therefore J is not an essential left ideal in $A[X]$, conversly, from (*) we have that $I = \psi(J) = \psi(Bf \cap A[x]) \cap A[X] \leq_e A[X]$, also we have $J \overset{\psi/J}{\to} I = \psi(J)$ is an isomorphism of left $A[X] - modules$.

Remarks 3.11. As a special case of [7], if I is a left (right, two sided) ideal in $A[X]$, then $C_n(I) = \{0 \neq a \in A : \exists P = aX^n + a_{n-1} \cdot X^{n-1} + \cdots + a_0,\ P \in I\} U \{0\}$ is a left (right, two sided) ideal in A for each $n \in N$. Moreover $C_n(I) \subset C_{n+1}(I)$ for each $n \in N$, which implies that $C(I) = \cup_{n \in N}, C_n(I)$ is a left (right, two sided) ideal in A. Notice that, if I is an essential left (right, two sided) ideal in $A[X]$, then $C_n(I)$ so is. But if A is a nonsinguar ring and I is a eft (right) ideal in $A[X]$ such that $C_0(I)$ is an essential in A, then I is an essential in $A[X]$. Moreover the converse is not true. Finally, if A is a semiprime ring and I is a two sided ideal in $A[X]$, then I is an essential in $A[X]$ if and only if $C[I]$ is an essential in A.

Now let A be a regular left seft-injective ring and J be a left ideal in $A[X]$ such that $C(J) = \cup_{n\in N} C_n(J) \leq_e A$.

We shall construct a left ideal J^* in $A[X]$ such that $J^* \subseteq J$ as follows. Since $C_0(J)$ is a left ideal in A, there exists an idempotent e_0 and orthogonal idempotents $\{f_{i_0}\}_{i_0\in I_0}$ in A such that $\oplus_{i_0\in I_0} Af_{i_0} \leq_e C_0(J) \leq_e Ae_0$. Also since $C_0(J) \subseteq C_1(J)$, there exists an idempotent e_1 and orthogonal idempotents $\{f_{i_1}\}_{i_1\in I'_1}$ in A such that $I'_1 \cap I_0 = \phi$ and

$$(\bigoplus_{i_0\in I_0} Af_{i_0}) \oplus (\bigoplus_{i_1\in I'_1} Af_{i_1}) \leq_e C_1(J) \leq_e Ae_1$$

Similarly $C_0(J) \subseteq C_1(J) \subseteq C_2(J) \subseteq \cdots \subseteq C_n(J) \subseteq \cdots$ implies that $(\oplus_{i_0\in I_0} Af_{i_0}) \oplus (\oplus_{i_1\in I'_1} Af_{i_1}) \oplus \cdots \oplus (\oplus_{i_n\in I'_n} Af_{i_n}) \leq_e C_n(J) \leq_e Ae_n$ for each n. Note also that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ where $I_{n+1} = I_n \cup I'_{n+1}$ and $I_n \cap I'_{n+1} = \phi$, $n = 0, 1, 2, \cdots$

For each $i \in \cup_n I_n$, let $m = \min\{k \in N : i \in I'_k\}$ hence $i = I'_m \in I_m$, implies that $f_i = f_{i_m} \in C_m(J)$, therefore there exists $P = f_i X^m + a_{m-1}X^{m-1} + \cdots + a_0 \in J$, and we take $P_i = f_i P \in J$. Now we define the left ideal J^* as follows

$$J^* = \sum_{i\in\bigcup_n I_n} A[X]\, P_i$$

Proposition 3.12. J^* is an essential left ideal in J.

Proof. To prove that, let for each $p \in N$, J_p be that left ideal of $A[X]$ generated by the polynomials of J of degree $\leq p$ and J^*_p be the left ideal of $A[X]$ generated by the polynomials P_i constructed above with degree $\leq p$.

First. we shall prove that $J^*_0 \leq_e J_0$
Notice that $J_0 = A[X]C_0(J)$ and $J^*_0 = \oplus_{i_0\in I_0} A[X] f_{i_0}$. Let $f \in J_0$, then $f = b_n X^n + b_{n-1}X^{n-1} + \cdots + b_0$ with $0 \neq b_i \in C_0(J)$, $i = 0, 1, \cdots, n$, since $\oplus_{i_0\in I_0} A\, f_{i_0} \leq_e C_0(J)$ then for each b_i, there exists an essential left ideal K_i of A such that $0 \neq K_i\, b_i \subseteq \oplus_{i_0\in I_0} A\, f_{i_0}$. Since $K = \cap_{i=0}^n K_i \leq_e A$ and A is left and right nonsongular ring, then $0 \neq K\, b_i \subseteq \oplus_{i_0\in I_0} A\, f_{i_0}$ for each $i = 0, 1, \cdots, n$. Let $0 \neq \xi_i\, b_i \in \oplus_{i_0\in I_0} A\, f_{i_0}$, $\xi_i \in K$ for each $i = 0, 1, \cdots, n$, which implies that $0 \neq \xi_j\, f = \sum_{i=0}^n \xi_j\, b_i\, X^i \in \oplus_{i_0\in I_0} A[X] f_{i_0} = J^*_0$ for each $j = 0, 1, \cdots, n$. Therefore $J^*_0 \leq_e J_0$.

NOw we shall prove that $J^* \leq_e J$. Let $Q \in J$, we use the induction on the degree of Q. If degree $Q = 0$, then $Q = a \in C_0(J) \subseteq J_0$, since $J^*_0 \leq_e J_0$, we can find an essential left ideal K of $A[X]$ such that $0 \neq K\, a = K\, Q \subseteq J^*_0 \subseteq J^*$.
Suppose that this is true for any polynomial Q in J with degree less than or equal to $n - 1$. Let $Q = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in J$ with $a_n \neq 0$, hence $a_n \in C_n(J)$. Let $p_0 = \inf\{p \in N : a_n \in C_p(J)\}$, since $\oplus_{i_j} A\, f_{i_j} \leq_e C_{p_0}(J)$, where $i_j \in \cup_{\ell=0}^{p_0} I_\ell$, hence there

exists $\lambda \in A$ such that $0 \neq \lambda a_n \in \oplus_{i_j} A f_{i_j}$, $i_j \in \cup_{\ell=0}^{p_0} I_\ell$ Therefore $\lambda a_n = \sum_{j=0}^{k} a_{i_j}^* f_{i_j}$, where $i_j \in \cup_{\ell=0}^{p_0} I_\ell$, and we can assume that $i_0 < i_1 < i_2 \cdots < i_k < p_0$. Since $P_{i_j} = f_{i_j} X^{i_j} + f_{i_j} a_{i_{j-1}} X^{i_{j-1}} + \cdots f_{i_j} a_0 \in J^*$ with $P_{i_j} = f_{i_j} P_{i_j}$, hence $a_{i_j}^* P_{i_j} \in J^*$ which implies that $Q' = a_{i_k}^* P_{i_k} + a_{i_{k-1}}^* X^{i_k - i_{k-1}} P_{i_{k-1}} + \cdots + a_{i_0}^* X^{i_k - i_0} P_{i_0}$ is in J^* with leading coefficient $\sum_{j=0}^{k} a_{i_j}^* f_{i_j}$, where $i_j \in \cup_{\ell=0}^{p_0} I_\ell$. Therefore $Q_1 = \lambda Q - X^{n-i_k} Q'$ is of degree less than or equal to $n-1$, which implies the existance of an essential left ideal L of $A[X]$ such that $L Q' \subseteq J^*$, hence $L \lambda Q \subseteq J^*$. Let K be the complement of $\ell_{A[X]}(\lambda Q)$ in $A[X]$, since $A[X]$ is left and right nonsingular (See [6] lemma 3.1) then $\ell_{A[X]}(\lambda Q)$ is not essential in $A[X]$. Therefore K is a nonzero left ideal in $A[X]$, which implies that $L \cap K \neq 0$, hence there exists $0 \neq g \in L \cap K$ such that $0 \neq g \lambda Q \in L \lambda Q$. Therefore $0 \neq L \lambda Q \subseteq J^*$ which implies that J^* is an essential in J.

Now we shall define other left ideal J' in $A[X]$ such that J^* is an essential in J'as follows.

We have that $\oplus_{i_0 \in I_0} A f_{i_0} \leq_e C_0(J) \leq_e A e_0$, and since $\ell_A(e_0) = A(1-e_0)$, then $(\oplus_{i_0 \in I_0} A f_{i_0}) \oplus \ell_A(e_0) \leq_e A$ and we define a homomorphism of left A-modules

$$\varphi : (\bigoplus_{i_0 \in I_0} A f_{i_0}) \oplus \ell_A(e_0) \to A$$

such that φ is the identity on $\oplus_{i_0 \in I_0} A f_{i_0}$ and zero on $\ell_A(e_0)$, since A is a left self-injective, φ can be extended to a homomrphism $\varphi^* : A \to A$. Let $f_0 = \varphi^*(1)$ and put $P_0 = f_0$.

For each $i_1 \in I_1'$, $P_{i_1} = f_{i_1} X + f_{i_1} a_{i_1}^0$, we define two homomorphisms of left A-modules

$$\varphi \text{ and } \psi : (\bigoplus_{i_0 \in I_0} A f_{i_0}) \oplus (\bigoplus_{i_1 \in I_1'} A f_{i_1}) \oplus \ell_A(e_1) \to A$$

such that φ is the identity on $\oplus_{i_1 \in I_1} A f_{i_1}$ and zero otherwise, but $\psi(f_{i_1}) = a_{i_1}^0$ for each $i_1 \in I_1$ and zero on $(\oplus_{i_0 \in I_0} A f_{i_0}) \oplus \ell_A(e_1)$,similarly φ and ψ are extended to homomorphisms φ^* and ψ^*, we let $f_1 = \varphi^*(1)$, $a_1^0 = \psi^*(1)$ and define $P_1 = f_1 X + f_1 a_1^0$. In general for each $i_n \in I_n'$ we have that $P_{i_n} = f_{i_n} X^n + f_{i_n} a_{i_n}^{n-1} X^{n-1} + \cdots + f_{i_n} a_{i_n}^0$ we define homomorphisms of left A-modules $\varphi_i : (\oplus_{i_0 \in I_0} A f_{i_0}) \oplus (\oplus_{i_1 \in I_1'} A f_{i_1}) \oplus \cdots \oplus (\oplus_{i_n \in I_n'} A f_{i_n}) \oplus \ell_A(e_n) \to A$, $i = 0, 1, \cdots, n$ as follows φ_n is the identity homomorphisms at $\oplus_{i_n \in I_n} A f_{i_n}$ and zero otherwise, but $\varphi_j(f_{i_n}) = a_{i_n}^j$ for each f_{i_n} and for $j = 0, 1, \cdots, n-1$ and zero otherwise φ_n and φ_j, $j = 0, 1, \cdots, n-1$ can be extended to homomorphism φ_n^* and φ_j^*, $j = 0, 1, \cdots, n-1$ from A into A, we define $\varphi_j^*(1) = a_n^j$, $j = 0, \cdots, n-1$, $\varphi_n^*(1) = f_n$, also we define P_n as the following $P_n = f_n X^n + f_n a_n^{n-1} X^{n-1} + \cdots + f_n a_n^1 X + f_n a_n^0$ and we define $J' = \sum_{n \in N} A[X] P_n$, which is countably generated left ideal in $A[X]$.

Remarks 3.13. Notice that $f_{i_n} f_n = f_{i_n} \varphi_n^*(1) = \varphi_n(f_{i_n}) = f_{i_n}$ and $f_{i_n} a_n^j = f_{i_n} \varphi_j^*(1) = \varphi_j(f_{i_n}) = a_{i_n}^j$ for each $j = 0, 1, \cdots, n-1$, wich implies that $f_{i_n} P_n = P_{i_n}$ for each $n \in N$ and since $J^* = \sum_{i_j \in \cup_n I_n} A[X] P_{i_j}$, we obtain that $J^* \subseteq J'$.

(2) It is clear that $C(J') = \cup_{n \in N} C_n(J') = \sum_{n \in N} A f_n$ also $f_{i_n} = f_{i_n} \cdot f_n$, implies that $\oplus_{i_n \in \cup_n I_n} A f_{i_n} \subseteq C(J')$.

(3) $\oplus_{i_n \in \cup_n I_n} A\, f_{i_n} \leq_e C(J) \leq_e A$ and $\oplus_{i_n \in \cup_n I_n} A\, f_{i_n} \subseteq C(J') \subseteq A$ implies $C(J') \leq_e A$ and $\oplus_{i_n \in \cup_n I_n} A\, f_{i_n} \leq_e C(J')$.

Propsition 3.14. J^* is an essential left ideal in J'.

Proof. Let $Q = \sum_{j-1}^{k} Q_j\, P_{t_j} \in J'$, where $P_{t_j} = f_{t_j} X^{t_j} + f_{t_j} a_{t_j}^{t_j-1} X^{t_j-1} + \cdots + f_{t_j} a_{t_j}^{1} X + f_{t_j} a_{t_j}^{0}$ for each $j = 1, \cdots, k$ and assume that $Q_j = a_{m_j}^{j} X^{m_j} + a_{m_j-1}^{j} X^{m_j-1} + \cdots + a_0^j$, $j = 1, \cdots, k$. Now consider $\{a_q^j\}_{q=0,1,\cdots,m_j}^{j=1,2,\cdots,k}$, since this family is contained in A and $C(J)$ is an essential left ideal in A, we can find an essential left ideal K in A such that $0 \neq K\, a_q^j \subseteq C(J)$ for each $j = 1, 2, \cdots, k$, $q = 0, 1, \cdots, m_j$. Hence $0 \neq K\, a_q^j \subseteq C_s(J)$ for some $s \in N$,which implies the existance of $\beta_j \in A$ such that $0 \neq \beta_j Q_j \in (C_s(J))[X]$, $j = 1, \cdots, k$. Let $\beta_j Q_j = \sum_{q=0}^{m_j} b_q^j X^q$ with at least one element of $b_q^j \neq 0$, denote it by d_j, as happend before in proposition 3.12 we can find an essential left ideal L in A such that $L\, b_q^j \subseteq \oplus_{i_s \in I_s} A\, f_{i_s}$ and $0 \neq L\, d_j$ for each $j = 1, 2, \cdots, k$. If K_j is the complement of $\ell_A(d_j\, f_{t_j})$ in A, then $K_j \cap L \neq 0$, which implies the existance of $\gamma_j \in K_j \cap L$ such that $0 \neq \gamma_j(d_j\, f_{t_j})$. Therefore $0 \neq \gamma_j(\beta_j\, Q_j) \in (\oplus_{i_s \in I_s} A\, f_{i_s})[X]$ and $(\gamma_j(\beta_j\, Q_j))P_{t_j} \neq 0$ for each $j = 1, 2, \cdots, k$. But we have that $f_{i_n} \cdot f_n = f_{i_n}$ and $f_{i_j}\, f_n = 0$, $j \neq n$, moreover $f_{i_n}\, a_n^j = f_{i_n} a_{i_n}^j$ for each $i_j \in \cup_n I_n$. Therefore $(\gamma_j \beta_j Q_j) P_{t_j} = Q_j' P_{i_j}$, $Q_j' \in A[X]$ and P_{i_j} is one of the generators of J^*, for each $j = 1, \cdots, k$, which implies that $\gamma_j \beta_j Q \in J^*$ for each $j = 1, 2, \cdots, k$. Hence $J^* \leq_e J'$.

Theorem 3.15. *If A is a regular left self-injective ring and J is a left ideal in $A[X]$ such that $C(J)$ is an essential left ideal in A, then there exists a countably generated left ideal J' in $A[X]$ such that $C(J')$is an essential left ideal in A and if J' is an essential left ideal in $A[X]$, then J is an essential left ideal in $A[X]$.*

Problem 1. Is J' is an essential left ideal in $A[X]$.

Remark 3.16. Let A be a regular left self-injective ring and J be a left ideal of A. If the answer of problem 1 is affirmative, then $C(J)$ is an essential left ideal in A, implies that J is an essential left ideal in $A[X]$.

Problem 2. Let A be a regular left self-injective ring and I, J be left ideals in $A[X]$ such that $I \simeq J$ as a left $A[X]$-modules. Is $C(I) \simeq C(J)$ as left A-modules.

Remark 3.17. Let A be a regular left slef-injective ring. If the answer of problems 1 and 2 are affirmative, then every directly finite idempotent in A is a directly finite idempotent in $E(A[X])$.
Since if e is directly finite in A and $e\,E(A[X])\,e$ is directly infinite, then from theorem 3.10 there exist two left ideals I and J in $(e\,A\,e)[X]$ such that $I \simeq J$, $I \leq_e (e\,A\,e)[X]$ and $J \not\leq_e (e\,A\,e)[X]$ as left $(e\,A\,e)[X]$-modules, which implies that $C(I) \simeq C(J)$, $C(I) \leq_e e\,A\,e$ and $C(J) \not\leq_e e\,A\,e$ as a left $e\,A\,e$-modules, but $e\,A\,e$ is a left slef-injective, hence this is a contradiction with $e\,A\,e$ is a directly finite ring.
In particular for regular left self-injective ring A, A is a directly finite ring if and only if $E(A[X])$ is a directly finite ring.

Proposition 3.18. *If A is a regular left self-injective ring, then A contains abelian idempotents if and only if $E(A[X])$ contains abelian idempotents.*

Proof. Let $f \neq 0$ be an abelian idempotent in $E(A[X])$, hence $f\,E\,(A[X])\,f$ is an abelian regular ring and $I = E(A[X])\,f \cap A[X]$ is a nonzero left ideal in $A[X]$. Let $P \in I$, $P = e\,X^n + a_{n-1}X^{n-1} + \cdots + a_0$ with minimal length and $e = e^2 \in A^*$, hence we have that $A[X]P \simeq A[X]e$, which implies that $E(A[X]).\,e \simeq E(A[X]P) \subseteq E(A[X])f$. Therefore $E(A[X])e$ is an abelian regular ring, hence e is an abelian idempotent in A. Conversely if e is an abelian idempotent in A, then e is an abelian idempotent in $E(A[X])$, by using (2.10) and (2.14).

Remark 3.19. Let A be a regular left self-injective ring, if the answers of problems 1 and 2 are affirmative, then
(1)A is of type II if and only if $E(A[X])$ is of type II.
(2)A is of type III if and only if $E(A[X])$ is of type III.
(3)A is a purely infinite ring if and only if $E(A[X])$ is a purely infinite ring.

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