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SOME INEQUALITIES AMONG GENERALIZED DIVERGENCE MEASURES

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Abstract. Burbea and Rao [3,4] and Sgarro [11] established an inequality between two famous divergence measures i.e., Jeffreys [6] invariant function (J-divergence) and Sibson's [13] information radius (R-divergence). Taneja [14,16] generalized both these divergences (J and R) having two scalar parameters. In this paper, we have extended the inequality between the R and J-divergences for two parametric cases. For one parametric generalizations of R and J-divergences, the generalized inequalities are improved.

I. Introduction

Let

$$\Delta_n = \{p = (p_1, p_2, \ldots, p_n),\ p_i > 0,\ \sum_{i=1}^{n} p_i = 1\}, \qquad n \geq 2$$

be the set of all complete finite discrete probability distributions.

Sibson [13] introduced a measure of information for two probability distributions, called *information radius* given by

$$R(P||Q) = \sum_{i=1}^{n} \Big[\frac{p_i \mathrm{Ln} p_i + q_i \mathrm{Ln} q_i}{2} - [\frac{p_i + q_i}{2}]\mathrm{Ln}[\frac{p_i + q_i}{2}]\Big] \tag{1}$$

for all $P, Q \in \Delta_n$.

The measure (1) arises due to the concavity property of Shannon's entropy. Sometimes, it is called [3,4] the *Jensen difference divergence measure*. By simple calculations, we can write

$$R(P||Q) = \frac{1}{2}\Big[D(P||\frac{P+Q}{2}) + D(Q||\frac{P+Q}{2})\Big], \tag{2}$$

for all $P, Q \in \Delta_n$, where $D(P||U)$ is the well-known Kullback-Leibler's [7] directed divergence, given by

$$D(P||U) = \sum_{i=1}^{n} p_i \,\mathrm{Ln} \frac{p_i}{u_i} \tag{3}$$

for all $P, U \in \Delta_n$ with $U = \frac{(P+Q)}{2}$. A symmetric version of (3) known as *J-divergence* (ref. Jeffreys [6]; Kullback and Leibler [7]) is given by

$$J(P||Q) \; = \; D(P||Q) \, + \, D(Q||P). \tag{4}$$

for all $P, Q \in \Delta_n$.

Burbea and Rao [3,4] and Sgarro [11] established an inequality between the measures (1) and (4) given by

$$J(P||Q) \; \geq \; 4\, R(P||Q) \tag{5}$$

for all $P, Q \in \Delta_n$.

Recently, Taneja [16] presented one and two parametric generalizations of the measure (2). The one and two parametric generalizations of the measure (4) are already given before by Taneja [14]. Some parametric generalizations of the measures (1) and (4) are also studied by Burbea [1,2] and Burbea and Rao [3,4]. In the following two subsections we have specified some of these generalizations having one and two scalar parameters written in unified expressions.

1.1. Unified (r,s)-Jensen Difference Divergence Measures

For all $P, Q \in \Delta_n$, let us consider the following divergence measures:

$${}^tR_r^s(P||Q) \qquad (r \; > \; 0, \; t \; = \; 1 \text{ and } 2,) \tag{6}$$

defined by

$${}^1R_r^s(P||Q) = [2(s-1)]^{-1}\Big\{\Big[\sum_{i=1}^n p_i^r\,[\frac{p_i+q_i}{2}]^{1-r}\Big]^{\frac{s-1}{r-1}} + \Big[\sum_{i=1}^n q_i^r\,[\frac{p_i+q_i}{2}]^{1-r}\Big]^{\frac{s-1}{r-1}} - 2\Big\},$$

$${}^2R_r^s(P||Q) = (s-1)^{-1}\Big\{\Big[\sum_{i=1}^n [\frac{p_i^r+q_i^r}{2}]\,[\frac{p_i+q_i}{2}]^{1-r}\Big]^{\frac{s-1}{r-1}} - 1\Big\},$$

$$\text{when } r \neq 1, s \neq 1, r > 0,$$

with the boundary cases continuously extended by L'Hopital Rule:

$${}^1R_1^s(P||Q) = [2(s-1)]^{-1}\Big\{\exp_e\Big[(s-1)\sum_{i=1}^n p_i \mathrm{Ln}[\frac{2p_i}{p_i+q_i}]\Big]$$

$$+ \exp_e\Big[(s-1)\sum_{i=1}^n p_i \mathrm{Ln}[\frac{2p_i}{p_i+q_i}]\Big] - 2\Big\}, \qquad (s \neq 1)$$

$${}^2R_1^s(P||Q) = (s-1)^{-1}\Big\{\exp_e\Big[(s-1)\,R(P||Q)\Big] - 1\Big\}, \qquad (s \neq 1)$$

$${}^1R_r^1(P||Q) = [2(r-1)]^{-1}\mathrm{Ln}\Big\{\Big[\sum_{i=1}^n p_i^r\,[\frac{p_i+q_i}{2}]^{1-r}\Big]\Big[\sum_{i=1}^n q_i^r\,[\frac{p_i+q_i}{2}]^{1-r}\Big]\Big\},$$

$$(r \neq 1, r > 0)$$

and

$$^{2}R_r^1(P||Q) = (r-1)^{-1}\mathrm{Ln}\Big\{\sum_{i=1}^{n}\Big[\frac{p_i^r+q_i^r}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-r}\Big\},\ (r\neq 1, r>0).$$

We call them the *unified* (r,s)*-Jensen difference divergence measures*. In particular, when $r=s$, we have

$$\begin{aligned} ^{1}R_s^s(P||Q) &= {}^{2}R_s^s(P||Q) = R_s^s(P||Q) \\ &= (s-1)^{-1}\Big\{\sum_{i=1}^{n}\Big[\frac{p_i^s+q_i^s}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-s}-1\Big\}, \qquad s\neq 1, s>0 \qquad (7)\end{aligned}$$

It can easily be checked (ref. Taneja [16]) that the measures $^{t}R_r^s(P||Q)$ ($t=1$ and 2) are nonnegative for all $r>0$ and any s. For more properties of the measures given by (6) such as convexity, Schur-convexity, monotonicity with respect to parameters, data processing inequalities etc. refer to Menéndez et al. [8].

All the measures appearing in (6) are due to Taneja [16]. While, the measure (7) was studied before by Taneja [17]. For applications of the measures (6) to comparison of experiments, Fisher measure of information and statistical pattern recognition refer to [8] [9] [19].

1.2. Unified (r,s)-J-Divergence Measures

For all $P,Q\in\Delta_n$, let us consider the following divergence measures:

$$^{t}J_r^s(P||Q) \qquad (r>0,\ t=1 \text{ and } 2), \qquad (8)$$

defined by

$$\begin{aligned} ^{1}J_r^s(P||Q) &=(s-1)^{-1}\Big\{\Big[\sum_{i=1}^{n}p_i^r q_i^{1-r}\Big]^{\frac{s-1}{r-1}}+\Big[\sum_{i=1}^{n}p_i^{1-r}q_i^r\Big]^{\frac{s-1}{r-1}}-2\Big\}, \\ ^{2}J_r^s(P||Q) &=2(s-1)^{-1}\Big\{\Big[\sum_{i=1}^{n}\Big[\frac{p_i^r q_i^{1-r}+p_i^{1-r}q_i^r}{2}\Big]\Big]^{\frac{s-1}{r-1}}-1\Big\}, \\ &\text{when } r\neq 1, s\neq 1, r>0,\end{aligned}$$

with boundary cases:

$$\begin{aligned} ^{1}J_1^s(P||Q) &= (s-1)^{-1}\Big\{\exp_e\Big[(s-1)\sum_{i=1}^{n}p_i\mathrm{Ln}\frac{p_i}{q_i}\Big] \\ &\quad+\exp_e\Big[(s-1)\sum_{i=1}^{n}q_i\mathrm{Ln}\frac{q_i}{p_i}\Big]-2\Big\}, s\neq 1 \\ ^{2}J_1^s(P||Q) &= 2(s-1)^{-1}\Big\{\exp_e\Big[\Big(\frac{s-1}{2}\Big)J(P||Q)\Big]-1\Big\}, s\neq 1 \\ ^{1}J_r^1(P||Q) &= (r-1)^{-1}\mathrm{Ln}\Big\{\Big[\sum_{i=1}^{n}p_i^r q_i^{1-r}\Big]\Big[\sum_{i=1}^{n}q_i^r p_i^{1-r}\Big]\Big\}, r\neq 1, r>0\end{aligned}$$

and

$$^2J_r^1(P||Q) = 2(r-1)^{-1}\text{Ln}\Big\{\sum_{i=1}^{n}[\frac{p_i^r q_i^{1-r} + p_i^{1-r} q_i^r}{2}]\Big\}, \qquad r \neq 1, r > 0$$

We call them the *unified* (r,s)*-J-divergence measures.* In particular, when $r = s$, we have

$$\begin{aligned} {}^1J_s^s(P||Q) &= {}^2J_s^s(P||Q) = J_s^s(P||Q) \\ &= (s-1)^{-1}\Big\{\sum_{i=1}^{n}[p_i^s q_i^{1-s} + p_i^{1-s} q_i^s] - 2\Big\}, s \neq 1, s > 0 \end{aligned} \tag{9}$$

It can easily be checked (ref. Taneja [16]) that the measures ${}^tJ_r^s(P||Q)$ $(t = 1$ and $2)$ are nonnegative for all $r > 0$ and any s. For more properties of the measures given by (8) such as convexity, Schur-convexity, monotonicity with respect to parameters, generalized data processing inequalities etc. refer to Taneja et al. [18]. For applications of these measures (8) to comparison of experiments, Fisher measure of information and statistical pattern recognition refer to [15] [16].

Most of the measures appearing in the unified expression (8) are due to Taneja [16], except the measures ${}^1J_r^1(P||Q)$ and $J_s^s(P||Q)$. The measure ${}^1J_r^1(P||Q)$ is due to Burbea [2] and the measure $J_s^s(P||Q)$ is due to Burbea and Rao [3,4] and Rathie and Sheng [10].

In this paper our aim is to generalize the inequality (5) for the measures given by (6) and (8). Some mixed inequalities are also given.

II. Inequalities for generalized divergence measures.

In this section we generalize the inequality (5) for the measures (6) and (8). For one parametric generalizations of R and J, the general results are improved.

Theorem 1. *For all $P, Q \in \Delta_n$, we have*

$${}^tJ_r^s(P||Q) \geq 2\ {}^tR_r^s(P||Q) \qquad (t = 1 \text{ and } 2) \tag{10}$$

for all $r > 0$ and any s.

Proof.

Case $t = 1$.

Let $P = (p_1, \ldots, p_n) \in \Delta_n$ and $Q = (q_1, \ldots, q_n) \in \Delta_n$ be two probability distributions. Then by using Jensen inequality, we can write

$$\frac{p_i^{1-r} + q_i^{1-r}}{2} \begin{cases} \leq [\frac{p_i + q_i}{2}]^{1-r}, & 0 < 1 - r < 1 \\ \geq [\frac{p_i + q_i}{2}]^{1-r}, & 1 - r > 1 \text{ or } 1 - r < 0 \end{cases}$$

for all $i = 1, 2, \cdots, n$. Multiplying by p_i^r and summing over all $i = 1, 2, \cdots, n$, we get

$$\sum_{i=1}^{n} p_i^r \left[\frac{p_i^{1-r} + q_i^{1-r}}{2}\right] \begin{cases} \leq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & r > 1 \end{cases}$$

i.e.,

$$\frac{1}{2}\left[1 + \sum_{i=1}^{n} p_i^r q_i^{1-r}\right] \begin{cases} \leq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & r > 1 \end{cases} \tag{11}$$

Similarly, we can get

$$\frac{1}{2}\left[1 + \sum_{i=1}^{n} q_i^r p_i^{1-r}\right] \begin{cases} \leq \sum_{i=1}^{n} q_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} q_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & r > 1 \end{cases} \tag{12}$$

It is easy to check that

$$\sum_{i=1}^{n} p_i^r q_i^{1-r} \begin{cases} \leq 1, & 0 < r < 1 \\ \geq 1, & r > 1 \end{cases} \tag{13}$$

Also, it can easily be checked that if $A \leq 1$, then $A \leq \frac{A+1}{2}$ and if $A \geq 1$, then $A \geq \frac{A+1}{2}$. This gives,

$$\sum_{i=1}^{n} p_i^r q_i^{1-r} \begin{cases} \leq \frac{1}{2}\left[1 + \sum_{i=1}^{n} p_i^r q_i^{1-r}\right], & 0 < r < 1 \\ \geq \frac{1}{2}\left[1 + \sum_{i=1}^{n} p_i^r q_i^{1-r}\right], & r > 1 \end{cases} \tag{14}$$

From (11) and (14) we have

$$\sum_{i=1}^{n} p_i^r q_i^{1-r} \begin{cases} \leq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} p_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & r > 1 \end{cases} \tag{15}$$

Similarly, from (12) we can get

$$\sum_{i=1}^{n} q_i^r p_i^{1-r} \begin{cases} \leq \sum_{i=1}^{n} q_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} q_i^r \left[\frac{p_i + q_i}{2}\right]^{1-r}, & r > 1 \end{cases} \tag{16}$$

Raising both sides of (15) and (16) by $\frac{s-1}{r-1}$ $(r \neq 1,\ s \neq 1)$, and adding we get

$$[\sum_{i=1}^{n} p_i^r q_i^{1-r}]^{\frac{s-1}{r-1}} + [\sum_{i=1}^{n} q_i^r p_i^{1-r}]^{\frac{s-1}{r-1}}$$

$$\begin{cases} \leq [\sum_{i=1}^{n} p_i^r [\frac{p_i+q_i}{2}]^{1-r}]^{\frac{s-1}{r-1}} + [\sum_{i=1}^{n} q_i^r [\frac{p_i+q_i}{2}]^{1-r}]^{\frac{s-1}{r-1}}. & (0 < r \neq 1, s-1 < 0) \\ \geq [\sum_{i=1}^{n} p_i^r [\frac{p_i+q_i}{2}]^{1-r}]^{\frac{s-1}{r-1}} + [\sum_{i=1}^{n} q_i^r [\frac{p_i+q_i}{2}]^{1-r}]^{\frac{s-1}{r-1}}. & (0 < r \neq 1, s-1 > 0) \end{cases} \tag{17}$$

Subtracting 2 on both sides of (17), multiplying by $(s-1)^{-1}$ $(s \neq 1)$, and simplifying, we get

$$^1J_r^s(P||Q) \ \geq\ 2\ ^1R_r^s(P||Q) \tag{18}$$

for all $r > 0,\ s \neq 1,\ r \neq 1$.

Case $t = 2$.

Adding (11) and (12), we get

$$1 + \frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}] \begin{cases} \leq \sum_{i=1}^{n} (p_i^r + q_i^r)[\frac{p_i+q_i}{2}]^{1-r} & 0 < r < 1 \\ \geq \sum_{i=1}^{n} (p_i^r + q_i^r)[\frac{p_i+q_i}{2}]^{1-r} & r > 1 \end{cases} \tag{20}$$

Similar to (13), we can check that

$$\frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}] \begin{cases} \leq 1, & 0 < r < 1 \\ \geq 1, & r > 1 \end{cases} \tag{21}$$

Again using the fact that, if $A \leq 1$, then $A \leq \frac{A+1}{2}$ and if $A \geq 1$, then $A \geq \frac{A+1}{2}$, from (21), we get

$$\frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + p_i^{1-r} q_i] \begin{cases} \leq \{1 + \frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}]\}, & 0 < r < 1 \\ \geq \{1 + \frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}]\} & r > 1 \end{cases} \tag{22}$$

From (20) and (22), we get

$$\frac{1}{2}\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}] \begin{cases} \leq \sum_{i=1}^{n} [\frac{p_i^r + q_i^r}{2}][\frac{p_i+q_i}{2}]^{1-r}, & 0 < r < 1 \\ \geq \sum_{i=1}^{n} [\frac{p_i^r + q_i^r}{2}][\frac{p_i+q_i}{2}]^{1-r}, & r > 1 \end{cases} \tag{23}$$

Raising both sides of (23) by $\frac{s-1}{r-1}$ $(s \neq 1,\ r \neq 1)$, we get

$$\Big[\sum_{i=1}^{n}\Big[\frac{p_i^r q_i^{1-r}+p_i^{1-r}q_i^r}{2}\Big]\Big]^{\frac{s-1}{r-1}}$$

$$\begin{cases} \leq \Big[\sum_{i=1}^{n}\Big[\frac{p_i^r+q_i^r}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-r}\Big]^{\frac{s-1}{r-1}} & (0<r\neq 1, s-1<0) \\ \geq \Big[\sum_{i=1}^{n}\Big[\frac{p_i^r+q_i^r}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-r}\Big]^{\frac{s-1}{r-1}} & (0<r\neq 1, s-1<0) \end{cases} \tag{24}$$

Subtracting 1 on both sides of (24), multiplying by $(s-1)^{-1}$ $(s \neq 1)$ and simplifying we get

$$^2J_r^s(P||Q) \ \geq\ 2\ ^2R_r^s(P||Q) \tag{25}$$

for $r > 0,\ s \neq 1,\ r \neq 1$.

By continuous extensions (25) and (18) are valid for all $r > 0$ and any s.

In particular, when $r = s = 1$, then from theorem 1, we get

$$J(P||Q) \ \geq\ 2\,R(P||Q) \tag{27}$$

It is quite obvious that the inequality (5) is much better than the inequality (27). In the following theorem, we shall improve the results of theorem 1 in some particular cases i.e., when $r = s$ and $s = 1,\ r \neq 1$, and shall obtain the inequalities similar to (5).

Theorem 2. *For all $P, Q \in \Delta_n$, we have*

(i) $J_s^s(P||Q) \geq 4\,R_s^s(P||Q),\ s \neq 1,\ s > 0$

(ii) $^1J_r^1(P||Q) \geq 4\ ^1R_r^1(P||Q), \quad 0 < r < 1$

(iii) $^2J_r^s(P||Q) \geq 4\ ^2R_r^1(P||Q), \quad 0 < r < 1$

Proof. (i) Replacing r by s in (20), we have

$$1+\frac{1}{2}\sum_{i=1}^{n}\big[p_i^s\,q_i^{1-s}\ +\ q_i^s\,p_i^{1-s}\big]\begin{cases} \leq\ \sum_{i=1}^{n}(p_i^s\ +\ q_i^s)\Big[\frac{p_i\ +\ q_i}{2}\Big]^{1-s}, & 0\ <\ s\ <\ 1 \\ \geq\ \sum_{i=1}^{n}(p_i^s\ +\ q_i^s)\Big[\frac{p_i\ +\ q_i}{2}\Big]^{1-s}, & s\ >\ 1 \end{cases}$$

i.e.,

$$\sum_{i=1}^{n}\big[p_i^s q_i^{1-s}+q_i^s p_i^{1-s}\big]-1\begin{cases} \leq 2\Big[\sum_{i=1}^{n}\Big[\frac{p_i^s+q_i^s}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-s}-1\Big], & 0<s<1 \\ \geq 2\Big[\sum_{i=1}^{n}\Big[\frac{p_i^s+q_i^s}{2}\Big]\Big[\frac{p_i+q_i}{2}\Big]^{1-s}-1\Big], & s>1 \end{cases} \tag{28}$$

Multipliying both sides of (28) by $(s-1)^{-1}$ $(s \neq 1)$ and simplifying we get

$$\frac{J_s^s(P||Q)}{2} \geq 2\ R_s^s(P||Q), \qquad s\ \neq\ 1,\ s\ >\ 0$$

$$J_s^s(P||Q) \geq 4\ R_s^s(P||Q), \qquad s\ \neq\ 1,\ s\ >\ 0$$

This completes the proof of part (i).

(ii) We know that

$$(A^{\frac{1}{2}} - 1)^2 \geq 0 \qquad \text{for all } A \geq 0$$

this gives

$$A - 2A^{\frac{1}{2}} + 1 \geq 0$$

i.e.,

$$A + 1 \geq 2A^{\frac{1}{2}} \tag{29}$$

Let

$$A = \sum_{i=1}^{n} p_i^r q_i^{1-r}, \qquad r > 0$$

then from (29), we have

$$2\,[\sum_{i=1}^{n} p_i^r q_i^{1-r}]^{\frac{1}{2}} \leq 1 + \sum_{i=1}^{n} p_i^r q_i^{1-r}, \qquad r > 0 \tag{30}$$

Similarly, we can write

$$2\,[\sum_{i=1}^{n} q_i^r p_i^{1-r}]^{\frac{1}{2}} \leq 1 + \sum_{i=1}^{n} q_i^r p_i^{1-r}, \qquad r > 0 \tag{31}$$

From (11) and (30), we have

$$[\sum_{i=1}^{n} p_i^r q_i^{1-r}]^{\frac{1}{2}} \leq \sum_{i=1}^{n} p_i^r\,[\frac{p_i + q_i}{2}]^{1-r}, \qquad 0 < r < 1 \tag{32}$$

Again, from (12) and (31), we have

$$[\sum_{i=1}^{n} q_i^r p_i^{1-r}]^{\frac{1}{2}} \leq \sum_{i=1}^{n} q_i^r\,[\frac{p_i + q_i}{2}]^{1-r}, \qquad 0 < r < 1 \tag{33}$$

Taking $log(.)$ on both sides of (32) and (33) and adding, we get

$$\frac{1}{2}\mathrm{Ln}\{[\sum_{i=1}^{n} p_i^r q_i^{1-r}][\sum_{i=1}^{n} q_i^r p_i^{1-r}]\}$$
$$\leq \mathrm{Ln}\{[\sum_{i=1}^{n} p_i^r\,[\frac{p_i + q_i}{2}]^{1-r}][\sum_{i=1}^{n} q_i^r\,[\frac{p_i + q_i}{2}]^{1-r}]\}, \qquad 0 < r < 1 \tag{34}$$

Multiplying both sides of (34) by $(r-1)^{-1}$ $(r \neq 1)$, and simplifying we get

$$\frac{1}{2}\,{}^1J_r^1(P\|Q) \geq 2\,{}^1R_r^1(P\|Q),\ 0 < r < 1$$

i.e.,

$$ {}^1J_r^1(P||Q) \geq 4\, {}^1R_r^1(P||Q), \; 0 < r < 1 $$

This completes the proof of part (ii).

(iii) In (29), let us take

$$ A = \frac{1}{2} [\sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}], \qquad r > 0 $$

then from (12), we have

$$ 2 [\frac{1}{2} \sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}]]^{\frac{1}{2}} \leq 1 + \frac{1}{2} \sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}], \qquad r > 0 \quad (35) $$

from (20) and (35), we have

$$ [\frac{1}{2} \sum_{i=1}^{n} [p_i^r q_i^{1-r} + q_i^r p_i^{1-r}]]^{\frac{1}{2}} \leq \sum_{i=1}^{n} [\frac{p_i^r + q_i^r}{2}] [\frac{p_i + q_i}{2}]^{1-r}, \qquad 0 < r < 1, \quad (36) $$

Taking $log(.)$ on both sides of (36) and multiplying by $(r-1)^{-1}$ $(r \neq 1)$, we get

$$ {}^2J_r^1(P||Q) \geq 4\, {}^2R_r^1(P||Q), \qquad 0 < r < 1 $$

This completes the proof of part (iii). Hence, completes the theorem.

Theorem 3. *The following mixed inequalities hold:*

(a) *For $s \geq r$, we have*

(i) ${}^1J_r^s(P||Q) \geq {}^2J_r^s(P||Q) \geq 2\, {}^2R_r^s(P||Q)$,

(ii) ${}^1J_r^s(P||Q) \geq 2\, {}^1R_r^s(P||Q) \geq 2\, {}^2R_r^s(P||Q)$,

(b) *For $s \leq r$, we have*

(i) ${}^2J_r^s(P||Q) \geq {}^1J_r^s(P||Q) \geq {}^2R_r^s(P||Q)$,

(ii) ${}^2J_r^s(P||Q) \geq 2\, {}^2R_r^s(P||Q) \geq 2\, {}^1R_r^s(P||Q)$,

(c) *For $0 < r < 1$, we have*

(i) ${}^1J_r^1(P||Q) \geq {}^2J_r^1(P||Q) \geq 4\, {}^2R_r^1(P||Q)$,

(ii) ${}^1J_r^1(P||Q) \geq 4\, {}^1J_r^1(P||Q) \geq 4\, {}^2R_r^1(P||Q)$.

Proof. We known that (ref. Taneja [16])

$$ {}^1R_r^s(P||Q) \begin{cases} \leq {}^2R_r^s(P||Q), & s \leq r \\ \geq {}^2R_r^s(P||Q), & s \geq r \end{cases} \quad (37) $$

and

$$ {}^1J_r^s(P||Q) \begin{cases} \leq {}^2J_r^s(P||Q), & s \leq r \\ \geq {}^2J_r^s(P||Q), & s \geq r \end{cases} \quad (38) $$

for all $r > 0$ and any s.

In view of the theorems 1 and 2, and the inequalities (37) and (38), the proof follows immediately.

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