TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 2, Summer 1991

ON AN INTEGRAL INEQUALITY OF R. BELLMAN

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Abstract. We prove: If u and v are non-negative, concave functions defined on $[0,1]$ satisfying

$$\int_0^1 (u(x))^{2p}\,dx = \int_0^1 (v(x))^{2q}\,dx = 1, \qquad p > 0, \qquad q > 0,$$

then

$$\int_0^1 (u(x))^p (v(x))^q\,dx \geq \frac{2\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)} - 1.$$

1. Introduction.

The classical Cauchy-Schwarz inequality states that

$$\Big[\int_0^1 u(x)v(x)\,dx\Big]^2 \leq \int_0^1 (u(x))^2\,dx \int_0^1 (v(x))^2\,dx \tag{1.1}$$

is valid for all functions u and v which are integrable on $[0,1]$. Extensions and variants of this result were published, for instance, in the monographs [1] and [4]. It is natural to ask for a converse of inequality (1.1); this means: Does there exist a constant $K > 0$ such that

$$\Big[\int_0^1 u(x)v(x)\,dx\Big]^2 \geq K \int_0^1 (u(x))^2\,dx \int_0^1 (v(x))^2\,dx$$

holds?

In 1956 R. Bellman [2] (see also [1]) proved the following remarkable proposition which shows that under additional restrictions upon u and v the answer to this question is "yes":

Theorem 1. *If u and v are concave functions defined on $[0,1]$ normalized by*

$$\int_0^1 (u(x))^2\,dx = \int_0^1 (v(x))^2\,dx = 1 \tag{1.2}$$

and

$$u(0) = u(1) = 0, \qquad v(0) = v(1) = 0,$$

then

$$\int_0^1 u(x)v(x)\,dx \geq \frac{1}{2}. \tag{1.3}$$

The constant $\frac{1}{2}$ *is best possible.*

Interesting extensions of Bellman's theorem can be found in [1], [2] and [5].

The aim of this paper is to present a new generalization and refinement of inequality (1.3) by applying a method which is different from the one used by Bellman.

2. The main result.

We start with a lemma which may be of interest in itself.

Lemma. *Let* u*,* v *and* w *be integrable functions on* $[0,1]$ *satisfying*

$$\int_0^1 (u(x))^2\,dx = \int_0^1 (v(x))^2\,dx = 1,$$

and

$$\int_0^1 (w(x))^2\,dx \neq 0,$$

then

$$\frac{1}{2}\Big[1 + \int_0^1 u(x)v(x)\,dx\Big] \geq \frac{\int_0^1 u(x)w(x)\,dx \int_0^1 v(x)w(x)\,dx}{\int_0^1 (w(x))^2\,dx}. \tag{2.1}$$

Proof. Using Gram's determinant theorem [4] we get

$$\begin{vmatrix} \int_0^1 (u(x))^2 dx & \int_0^1 u(x)v(x)dx & \int_0^1 u(x)w(x)dx \\ \int_0^1 v(x)u(x)dx & \int_0^1 (v(x))^2 dx & \int_0^1 v(x)w(x)dx \\ \int_0^1 w(x)u(x)dx & \int_0^1 w(x)v(x)dx & \int_0^1 (w(x))^2 dx \end{vmatrix} \geq 0.$$

This leads to

$$\int_0^1 (w(x))^2 dx\Big[1 - \big[\int_0^1 u(x)v(x)dx\big]^2\Big] + 2\int_0^1 u(x)v(x)dx \int_0^1 v(x)w(x)dx \int_0^1 w(x)u(x)dx$$
$$\geq \big[\int_0^1 u(x)w(x)dx\big]^2 + \big[\int_0^1 v(x)w(x)dx\big]^2$$
$$\geq 2\int_0^1 u(x)w(x)dx \int_0^1 v(x)w(x)dx,$$

where the last inequality follows from the arithmetic mean - geometric mean inequality. Thus, we obtain

$$\int_0^1 (w(x))^2 dx \, [1 - \int_0^1 u(x)v(x)dx] \, [1 + \int_0^1 u(x)v(x)dx]$$
$$\geq 2 \int_0^1 u(x)w(x)dx \int_0^1 v(x)w(x)dx \, [1 - \int_0^1 u(x)v(x)dx].$$

From (1.1) we conclude $\int_0^1 u(x)v(x)dx \leq 1$, which yields

$$\int_0^1 (w(x))^2 \, dx \, [1 + \int_0^1 u(x)v(x)\, dx] \;\geq\; 2 \int_0^1 u(x)w(x)\, dx \int_0^1 v(x)w(x)\, dx;$$

which we had to show.

Remarks.

1. Inequality (2.1) contains the Cauchy-Schwarz inequality as a special case. Indeed, set $u = v$ and let $\int_0^1 (u(x))^2 dx = c^2 \neq 0$, then we have $\int_0^1 (\frac{u(x)}{c})^2 dx = 1$ and (2.1) becomes

$$\int_0^1 (u(x))^2 \, dx \int_0^1 (w(x))^2 \, dx \;\geq\; [\int_0^1 u(x)w(x)\, dx]^2. \tag{2.2}$$

2. An application of the Lemma and of Tchebyschef's inequality [4] leads to the following sharpening of (2.2) with $u(x) \equiv 1$:
If w is a positive, integrable function and if v is a positive, monotonic function on $[0, 1]$ satisfying $\int_0^1 (v(x))^2 dx = \int_0^1 (v(x))^{-2} dx = 1$, then

$$[\int_0^1 w(x)\, dx]^2 \;\leq\; \int_0^1 w(x)v(x)\, dx \int_0^1 \frac{w(x)}{v(x)}\, dx \;\leq\; \int_0^1 (w(x))^2 \, dx.$$

Next we establish a new extension of inequality (1.3).

Theorem 2. *If u and v are non-negative, concave functions defined on $[0, 1]$ satisfying*

$$\int_0^1 (u(x))^{2p} \, dx \;=\; \int_0^1 (v(x))^{2q} \, dx \;=\; 1, \qquad p > 0, \qquad q > 0,$$

then

$$\int_0^1 (u(x))^p (v(x))^q \, dx \;\geq\; \frac{2\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)} - 1. \tag{2.3}$$

Proof. If we replace in the Lemma u by u^p and v by v^q, and if we set $w(x) \equiv 1$, then inequality (2.1) becomes

$$\frac{1}{2}[1 + \int_0^1 (u(x))^p (v(x))^q \, dx] \;\geq\; \int_0^1 (u(x))^p \, dx \int_0^1 (v(x))^q \, dx. \tag{2.4}$$

Next we make use of following inequality due to L. Berwald [3], which is valid for non-negative, concave functions u:
If $0 < \alpha < \beta$, then

$$\left[(\beta+1)\int_0^1 (u(x))^\beta\,dx\right]^{\frac{1}{\beta}} \le \left[(\alpha+1)\int_0^1 (u(x))^\alpha\,dx\right]^{\frac{1}{\alpha}}. \tag{2.5}$$

From (2.5) with $\alpha = p$ and $\beta = 2p$ we conclude:

$$\left[(2p+1)\int_0^1 (u(x))^{2p}\,dx\right]^{\frac{1}{2p}} \le \left[(p+1)\int_0^1 (u(x))^p\,dx\right]^{\frac{1}{p}}$$

which leads to

$$\int_0^1 (u(x))^p\,dx \ge \frac{\sqrt{2p+1}}{p+1}. \tag{2.6}$$

Similarly, we obtain

$$\int_0^1 (v(x))^q\,dx \ge \frac{\sqrt{2q+1}}{q+1}, \tag{2.7}$$

such that the inequalities (2.4), (2.6) and (2.7) yield

$$\frac{1}{2}[1+\int_0^1 (u(x))^p(v(x))^q\,dx] \ge \frac{\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)},$$

which proves Theorem 2.

Remarks.

1. Inequality (2.3) is in particular true if u and v are concave functions defined on $[0,1]$ satisfying $\int_0^1 (u(x))^{2p}dx = \int_0^1 (v(x))^{2q}dx = 1$ and the boundary conditions $u(0) = u(1) = 0$ and $v(0) = v(1) = 0$.

2. We have also proved the following refined version of Bellman's inequality:
If u and v are non-negative, concave functions defined on $[0,1]$ such that $\int_0^1 (u(x))^2dx = \int_0^1 (v(x))^2dx = 1$, then

$$\int_0^1 u(x)v(x)\,dx \ge 2\int_0^1 u(x)\,dx\int_0^1 v(x)\,dx - 1 \ge \frac{1}{2}.$$

It might be surprising that a generalization of the Cauchy - Schwarz inequality (the Lemma) plays a central role in a proof for Bellman's converse of this inequality.

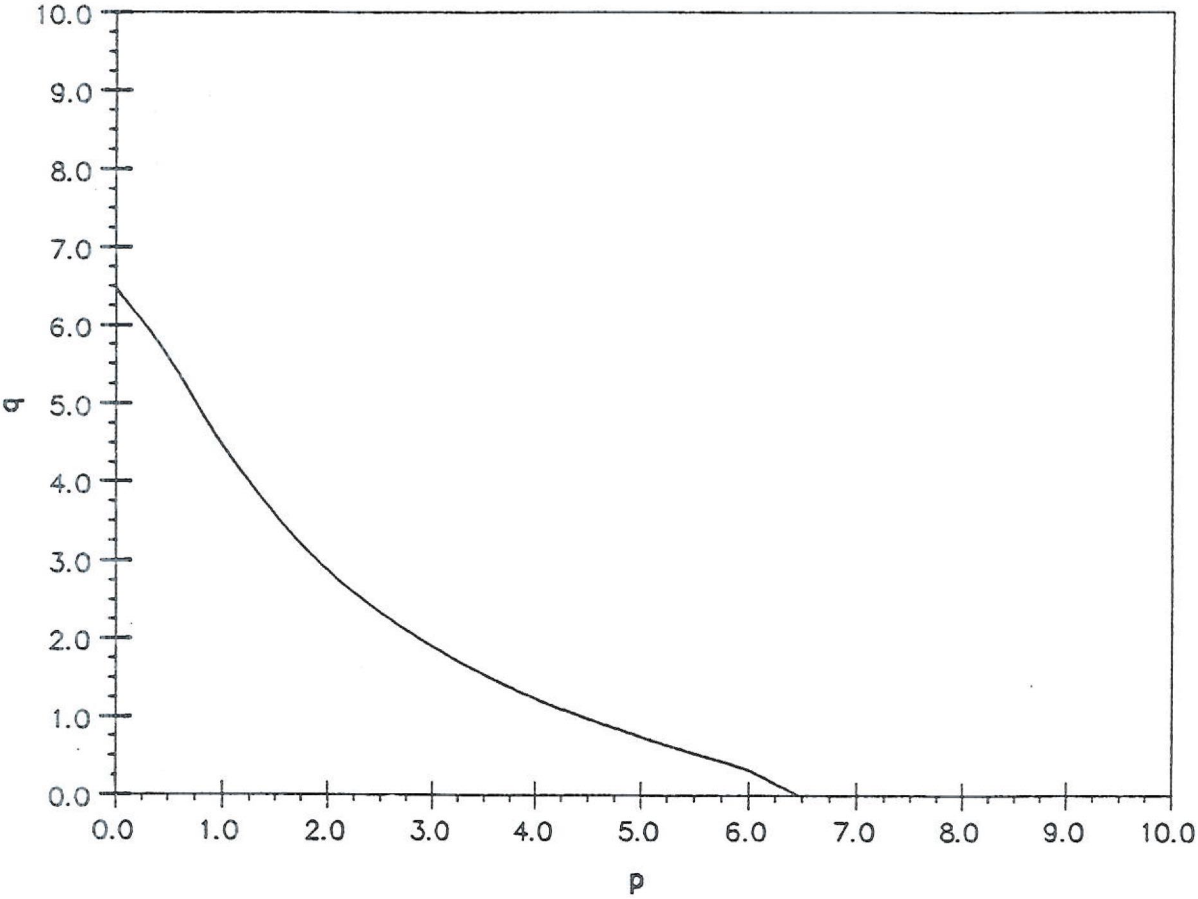
3. Since the integral $\int_0^1 (u(x))^p(v(x))^q dx$ is non-negative, inequality (2.3) is of interest only for positive values p and q such that

$$\frac{2\sqrt{(2p+1)(2q+1)}}{(p+1)(q+1)} - 1 > 0. \tag{2.8}$$

A simple calculation yields that (2.8) is fulfilled for all points (p, q) of the first quadrant lying below the curve

$$q = \frac{-p^2 + 6p + 3 + 4\sqrt{(2p+1)(-p^2+6p+3)}}{(p+1)^2}.$$

The graph of this curve is shown in the figure below.



Acknowledgement. I am very grateful to Professor W.S. Verwoerd for providing the figure and to Frank Bullock for helpful discussions.

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