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LYAPUNOV INEQUALITY FOR SYSTEM DISCONJUGACY OF EVEN ORDER DIFFERENTIAL EQUATIONS

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With respect to the following even order self-adjoint differential equation of order $2n$

$$(-1)^{n+1}x^{(2n)} + p(t)x = 0, \tag{1}$$

where $p(t)$ is continuous in $(-\infty, \infty)$, two points a and b $(a < b)$ in an interval I are conjugate if there is a nontrivial solution $x(t)$ of (1) which has n-fold zeros at a and b. Equation (1) is said to be disconjugate on I if no two points of the interval I are conjugate. Various disconjugacy criteria were obtained. In particular, a classic result of Lyapunov-Beurling-Borg-Wintner [1,3] states that when $n = 1$, the Lyapunov inequality

$$(b - a)\int_a^b |\, p(t)\,| \leq 4$$

is a disconjugacy criterion for (1) on the interval $[a, b]$. For general n, Levin [4] and Reid [5] stated that

$$(b - a)^{2n-1}\int_a^b |\, p(t)\,| \leq 4^{2n-1}(2n - 1)[(n - 1)!]^2 \tag{2}$$

is a disconjugacy criterion for equation (1) in $[a, b]$.

The concept of conjugate points defined above is partly originated from boundary value problem involving equation (1) and boundary conditions of the form

$$x^{(k)}(a) = 0 = x^{(k)}(b), \qquad k = 0, 1, \cdots, n.$$

However, it is equally important to consider boundary conditions of the form

$$x^{(2k)}(a) = 0 = x^{(2k)}(b), \qquad k = 0, 1, \cdots, n-1 \tag{3}$$

in practice. In this note, we shall define the concept of system conjugate points originated from the above mentioned boundary conditions and derive a Lyapunov inequality which serves as a disconjugacy criterion similar to (2).

More specifically, we say that two points a and b $(a < b)$ in an interval are system conjugate if (1) has a nontrivial solution $x(t)$ which satisfies (3). We say that equation (1) is system disconjugate on an interval if no two points of I are system conjugate.

Assume that $x(t)$ is a nontrivial solution of (1) which satisfies (3). Then by means of the Green's function $G_n(t, s|a, b)$ for the system

$$(-1)^n x^{(2n)}(t) = \delta(t - s) \tag{4a}$$
$$x^{(2k)}(a) = 0x^{(2k)}(b), \qquad k = 0, 1, \cdots, n-1 \tag{4b}$$

where δ is the Dirac delta function, $x(t)$ must satisfy the integral equation

$$x(t) = \int_a^b G_n(t, s|a, b)p(s)x(s)ds.$$

It is well known that

$$G_n(t, s \mid a, b) = \begin{cases} (b-t)(s-a)/(b-a) & a \le s \le t \le b \\ (t-s)(b-s)/(b-a) & a \le t \le s \le b \end{cases} \tag{5}$$

and

$$G(t, s|a, b) = \int_a^b G_1(t, r|a, b)G_{n-1}(r, s|a, b)dr, \qquad n = 2, 3, \cdots \tag{6}$$

By (5) and (6), it is clear that $G_n(t, s|a, b)$ is continuous on $[a, b] \times [a, b]$ and positive in the interior of $[a, b] \times [a, b]$. We shall need to find the maximum of $G_n(t, s|a, b)$ over $[a, b] \times [a, b]$. In order to do this, we define [2] a sequence of polynomials $f_1,\ f_2,\ f_3,\ \cdots$ by means of the conditions

$$f_1(x) = \frac{x}{2} \tag{7}$$
$$f_n'(x) = f_{n-1}(x), \qquad n > 1 \tag{8}$$
$$f_{2n-1}(-1) = 0, \qquad n > 1 \tag{9}$$
$$f_{2n}(x) = f_{2n}(-x), \qquad n \ge 1. \tag{10}$$

Denote the points $(-1, -1)$, $(0, 0)$, $(1. -1)$ and $(0, -2)$ by A, B, C and D respectively, and denote the parallelogram with vertices A, B, C and D by p. Let $H_n(u, v)$ be the function on P defined by

$$H_n(u, v) = \begin{cases} (-1)^n[f_{2n}(u) - f_{2n}(v)] & \text{if } (u, v) \in \Delta ABC \\ (-1)^n[f_{2n}(u) - f_{2n}(-v-2)] & \text{if } (u, v) \in \Delta ADC \end{cases} \tag{11}$$

Under the change of variables

$$t = (u - v)/2, \qquad s = (u + v + 2)/2, \qquad u = t + s - 1, \qquad v = s - t - 1,$$

it is easily seen that P is transformed into $[0,1] \times [0,1]$. We may further verify by means of the uniqueness conditions [2] for Green's function that

$$G_n(t, s|0,1) = H(t+s-1, s-t-1), \qquad (t,s) \in [0,1] \times [0,1] \tag{12}$$

and

$$G_n(t, s|a,b) = (b-a)^{2n-1} G_n\left(\frac{t-a}{b-a}, \frac{s-a}{b-a} \mid 0,1\right), \qquad (t,s) \in [0,1] \times [0,1]. \tag{13}$$

In view of (13), we see that the maximum of $G_n(t,s|a,b)$ is an increasing function of b. Indeed, suppose $b < b'$. Then for each $(t,s) \in [a,b] \times [a,b]$, letting

$$t' = a + \frac{b'-a}{b-a}(t-a), \qquad s' = a + \frac{b'-a}{b-a}(s-a),$$

we have $(t', s') \in [a, b'] \times [a, b']$ and

$$\begin{aligned} G_n(t, s \mid a, b) &= (b-a)^{2n-1} G_n\left(\frac{t-a}{b-a}, \frac{s-a}{b-a} \mid 0,1\right) \\ &< (b'-a)^{2n-1} G_n\left(\frac{t'-a}{b'-a}, \frac{s'-a}{b'-a} \mid 0,1\right) \\ &= G_n(t', s' \mid a, b'). \end{aligned}$$

Similarly, we can show that $G_n(t,s|a,b)$ is a decreasing function of a.

We can show further that the maximum of $G_n(t,s|a,b)$ over $[a,b] \times [a,b]$ is equal to $G_n(\frac{a+b}{2}, \frac{a+b}{2}|a,b)$. To see this, it suffices to show that the maximum of $H_n(u,v)$ is $H_n(0,-1)$. First observe by means of (10) and (11) that $H_n(u,v)$ is symmetric with respect to the line $u=0$ and the line $v=-1$. Let E be the point of intersection of the lines $u=0$ and $v=-1$. Then the maximum of $H_n(u,v)$ is equal to the maximum of $H_n(u,v)$ over the triangle ΔABE. It is known [2] that $(-1)^n f_{2n}$ is a strictly increasing function on $[-1,0]$, thus for any $(u,v) \in \Delta ABE$, we have

$$\begin{aligned} H_n(0,-1) &= (-1)^n [f_{2n}(0) - f_{2n}(-1)] \\ &> (-1)^n [f_{2n}(0) - f_{2n}(v)] \\ &> (-1)^n [f_{2n}(u) - f_{2n}(v)] \\ &= H_n(u,v) \end{aligned}$$

as required.

We can now calculate systematically the maximum of $H_n(u,v)$. First, we deduce from (7)-(10) that

$$f_{2n}(x) = c_0 x^{2n} + c_1 x^{2n-2} + \cdots + c_n x^2 + c_{n+1} \tag{14}$$

where $\text{col}(c_0, c_1, \ldots, c_n)$ is the solution of

$$\begin{bmatrix} 2n! & 0 & 0 & \cdots & 0 \\ 2n!/3! & (2n-2)! & 0 & \cdots & 0 \\ 2n!/5! & (2n-3)!/3! & (2n-4)! & & \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & & \\ 2n!/(2n-1)! & (2n-2)!/(2n-3)! & (2n-4)!/(2n-5)! & \cdots & 2! \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \tag{15}$$

Since

$$H_n(0,-1) = (-1)^n[f_{2n}(0) - f_{2n}(-1)] = (-1)^{n+1}[c_0 + c_1 + \ldots, + c_n],$$

we need to find $c_0 + c_1 + \ldots + c_n$ systematically. To do this, we let

$$\begin{aligned} A_0 &= (2n!), \qquad A_1 = \begin{bmatrix} 2n! & 0 \\ 2n!/3! & (2n-2)! \end{bmatrix}, \ldots, \\ A_n &= \text{the coefficient matrix in (15).} \end{aligned} \tag{16}$$

Also, let

$$\begin{aligned} M_0 &= 1, \qquad M_1 = -2n!/3!, \cdots \\ M_k &= \text{cofactor of the } (1, 1+k) - \text{element of } A_k. \end{aligned} \tag{17}$$

Then it is easily seen by induction that

$$c_0 + c_1 + \ldots + c_n = \frac{1}{2}\left\{\frac{M_0}{\det A_0} + \frac{M_1}{\det A_1} + \cdots + \frac{M_n}{\det A_n}\right\}. \tag{18}$$

Since M_0, $M_1, \ldots, M_n$ are integers and $\det A_0 = 2n!$, $\det A_1 = 2n!(2n-2)!, \ldots, \det A_n = 2n!(2n-1)! \cdots 2!$, taking common denominators $2n!(2n-2)! \cdots 2!$ of the fractions in (18), we see that $c_0 + c_1 + \ldots + c_n$ is a rational number. As examples,

$$H_1(0,-1) = 1/4, \ H_2(0,-1) = 1/48, \ H_3(0,-1) = 1/480.$$

We are now ready to prove the following result.

Theorem. *Let $A_0, A_1, \ldots, A_n$ and $M_0, M_1, \ldots, M_n$ be defined by (16) and (17) respectively. If*

$$(b-a)^{2n-1} \int_a^b |\, p(t) \,| \leq 2(-1)^{n+1}\left\{\frac{M_0}{\det A_0} + \frac{M_1}{\det A_1} + \ldots + \frac{M_n}{\det A_n}\right\}^{-1}, \tag{19}$$

then equation (1) is system disconjugate on $[a, b]$.

Proof. Suppose a' and b' $(a \leq a < b' \leq b)$ are system conjugate points in $[a, b]$, and $x(t)$ is a solution of (1) such that $x^{(2k)}(a') = 0 = x^{(2k)}(b')$ for $k = 0, 1, 2, \cdots, n-1$, then

$$x(t) = \int_{a'}^{b'} G_n(t, s \mid a', b')p(s)x(s)ds.$$

Let $x_{\max} = \max\{\mid x(t) \mid : a' \leq t \leq b'\}$, we will have

$$x_{\max} < x_{\max} \max G_n(t, s \mid a', b') \int_{a'}^{b'} \mid p(s) \mid ds,$$

so that

$$\int_a^b \mid p(s) \mid ds \geq \int_{a'}^{b'} \mid p(s) \mid ds > \frac{1}{\max G_n(t, s \mid a', b')} \geq \frac{1}{\max G_n(t, s \mid a, b)}$$

as required. *Q.E.D.*

There is a final remark we can make. If we let $x(t) = G_n(t, (a+b)/2|a, b)$ and $p(t)$ be the generalized function defined by

$$p(t) = \frac{(-1)^n x^{(2n)}(t)}{x(t)} = \frac{\delta(t - (a+b)/2)}{x(t)},$$

then $x(t)$ is a solution of (1) and satisfies (3), furthermore,

$$\int_a^b \mid p(s) \mid ds = \int_a^b \frac{\delta(t - (a+b)/2)}{G_n(t, (a+b)/2 \mid a, b)} dt = \frac{1}{G_n((a+b)/2, (a+b)/2 \mid a, b)}.$$

This shows that the inequality (19) is sharp, for we can use the standard approximation technique to construct sequences of continuous functions $\{x_j(t)\}$ and $\{p_j(t)\}$ such that

$$(-1)^{n+1} x_j^{(2n)} + p_j(t)x_j = 0, \qquad a \leq t \leq b$$
$$x_j^{(2k)}(a) = 0 = x_j^{(2k)}(b), \qquad k = 0, 1, \cdots, n-1,$$

and

$$\int_a^b \mid p_j(s) \mid ds \to \frac{1}{G_n((a+b)/2, (a+b)/2 \mid a, b)}.$$

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