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ON A CONSEQUENCE OF MILIN'S INEQUALITY FOR FABER POLYNOMIALS

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1. Introduction

Let S denote the class of all functions $f(z)$ which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$ and are normalized so that $f(0) = 0$ and $f'(0) = 1$. The logarithmic coefficients γ_k of $f(z)$ are defined by the relation

$$\log\left[\frac{f(z)}{z}\right] = 2\sum_{1}^{\infty}\gamma_k z^k$$

Louis de Branges [1, p. 146-150] proved the following inequalities, originally conjectured by Milin.

Theorem 1. *Let $f \in S$ and let γ_k $(k = 1, 2, \cdots)$ be the logarithmic coefficients of f. Then, for every $n \geq 1$, we have*

$$\sum_{k=1}^{n} k(n+1-k) \mid \gamma_k \mid^2 \leq \sum_{k=1}^{n} \frac{n+1-k}{k} \tag{1}$$

Equality holds if and only if $f(z) = z/(1-\eta z)^2$, $|\eta| = 1$.

In this paper, we shall show that this theorem easily implies a consequence for Faber polynomials, which suggests in turn a reasonable conjecture for them.

2. Faber Polynomials and the Class S_p

For $f \in S$, the functions $F_k(t)$ generated by the relation

$$\log\left[\frac{f(z)}{z(1-tf(z))}\right] = \sum_{k=1}^{\infty}\frac{1}{k}F_k(t)\, z^k$$

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for z in a neighborhood of the origin (depending on t) are called the Faber Polynomials of $f(z)$.

We prove the following result:

Theorem 2. *Let $f \in S$. If $F_k(t)$ are the Faber polynomials of f, and ω does not belong to the range of f, then*

$$\sum_{k=1}^{N} \frac{N+1-k}{k} \mid F_k(\frac{1}{\omega}) \mid^2 \leq 4 \sum_{k=1}^{N} \frac{N+1-k}{k} \tag{2}$$

Equality holds if and only if $f(z)$ has the form

$$f(z) = \frac{z}{1 - 2\alpha\eta z + \eta^2 z^2}$$

for some η, $|\eta| = 1$, and $\alpha \in [-1, +1]$.

Proof. Since $\omega \notin f(U)$, the function $g_\omega(z) = \frac{\omega f(z)}{(\omega - f(z))}$ belongs to the class S. Since

$$\log\left[\frac{g_\omega(z)}{z}\right] = \log\left[\frac{f(z)}{z(1 - \frac{f(z)}{\omega})}\right] = 2\sum_{k=1}^{\infty}\left[\frac{1}{2k}F_k(\frac{1}{\omega})\right] z^k$$

the logarithmic coefficients of $g_\omega(z)$ are determined. Applying Theorem 1 to g_ω, we obtain (2) above. Equality holds if and only if

$$g_\omega(z) = \frac{\omega f(z)}{\omega - f(z)} = \frac{z}{(1 - \eta z)^2}$$

for some η, $|\eta| = 1$. Such an $f(z)$ must have the form

$$f(z) = \frac{z}{1 - (2\eta - \frac{1}{\omega})z + \eta^2 z^2}.$$

Now, the product of the roots of the quadratic in the denominator has modulus equal to one; hence, both roots must be on the unit circle. If we denote these roots by $z_1 = \overline{\eta} e^{i\theta}$ and $z_2 = \overline{\eta} e^{-i\theta}$, then

$$\begin{aligned} 1 - (2\eta - \frac{1}{\omega})z + \eta^2 z^2 &= (1 - \overline{z}_1 z)(1 - \overline{z}_2 z) \\ &= 1 - 2\eta \cos\theta\, z + \eta^2 z^2 \end{aligned}$$

showing that the extremal function has specified form. This completes the proof of the theorem.

Remark. If $\omega \to \infty$ along some continuum, then $F_k(\frac{1}{\omega}) \to F_k(0) = 2k\gamma_k$; thus, the inequality of Theorem 2 implies the inequality of Theorem 1.

Now let $0 < p < 1$. Suppose that we wish to maximize the nonlinear functional

$$f \to \sum_{k=1}^{N} \frac{N+1-k}{k} \mid F_k(\frac{1}{f(p)}) \mid^2$$

over the class S. Taking the Theorem 2 into consideration, the natural candiate for an extremal function would be given by

$$\hat{f}(z) = \frac{z}{1 - 2\alpha\eta z + \eta^2 z^2}$$

where $|\eta| = 1$ and $\alpha \in [-1, +1]$. On one hand, for any function $f \in S$, the definition of Faber Polynomials gives us the relation

$$\log\left[\frac{f(z)}{z(1 - \frac{f(z)}{f(p)})}\right] = \sum_{k=1}^{\infty} \frac{1}{k} F_k(\frac{1}{f(p)})\, z^k$$

where we have set $t = 1/f(p)$. On the other hand, for the function $\hat{f}(z)$, a short computation yields

$$\log\left[\frac{\hat{f}(z)}{z(1 - \frac{\hat{f}(z)}{\hat{f}(p)})}\right] = \sum_{k=1}^{\infty} \frac{1}{k} (p^k \eta^{2k} + \frac{1}{p^k})\, z^k$$

Since $|p^k\eta^{2k} + \frac{1}{p^k}| = p^k + \frac{1}{p^k}$ only if $\eta = \pm 1$, we are naturally led to the following conjecture.

Conjecture I. Let $f \in S$ and $p \in (0,1)$. If $F_k(t)$ $(k = 1, 2, \cdots)$ are the Faber polynomials of $f(z)$, then, for each $N \geq 1$, we have

$$\sum_{k=1}^{N} \frac{N+1-k}{k} \mid F_k(\frac{1}{f(p)}) \mid^2 \leq \sum_{k=1}^{N} \frac{N+1-k}{k} (p^k + \frac{1}{p^k})^2. \tag{3}$$

Equality holds if and only if

$$f(z) = \frac{z}{1 - 2\alpha z + z^2} \qquad (\alpha \in [-1, +1])$$

Remark. These conjectured inequalities may be viewed as an extension of the inequalities of Theorem 2. To see this, let $p_n = 1 - \frac{1}{n}$ $(n = 1, 2, \cdots)$ and let $f \in S$. If the sequence $f(p_n)$ $(n = 1, 2, \cdots)$ is unbounded, then there must exist a subsequence $f(p_{n_j})$ $(j = 1, 2, \cdots)$ such that $f(p_{n_j}) \to \infty$ as $j \to +\infty$. But then $F_k(\frac{1}{f(p_{n_j})}) \to F_k(0) = 2k\gamma_k$. If the sequence $f(p_n)$ is bounded, then there must exist a subsequence $f(p_{n_j})$ $(j = 1, 2, \cdots)$ which converges to some finite boundary point ω. But then we would have $F_k(\frac{1}{f(p_{n_j})}) \to F_k(\frac{1}{\omega})$ as $j \to +\infty$. Replacing p with p_{n_j} in (3) and then letting $j \to \infty$, we obtain the inequality of Theorem 2, for some particular boundary point (∞ or ω). Thus, Conjecture I implies inequalities which are known to be true.

Now let S_p $(0 < p < 1)$ denote the class of meromorphic univalent functions $g(z)$ defined on the unit disk U and normalized so that $g(0) = 0$, $g'(0) = 1$ and $g(p) = \infty$. We define the logarithmic coefficients of $g(z) \in S_p$ for z near the origin by the relation

$$\log\left[\frac{g(z)}{z}\right] = 2\sum_{k=1}^{\infty} \delta_k z^k$$

We observe that the logarithmic coefficients of any $g \in S_p$ may be expressed in terms of the Faber Polynomials of some $f \in S$. Indeed, for any $f \in S$, the function

$$g(z) = \frac{f(p)f(z)}{f(p) - f(z)} \tag{4}$$

belongs to S_p, and

$$\log\left[\frac{g(z)}{z}\right] = \log\left[\frac{f(z)}{z(1 - \frac{f(z)}{f(p)})}\right] = 2\sum_{k=1}^{\infty} \frac{1}{2k} F_k(\frac{1}{f(p)})\, z^k.$$

Conversely, for any $g \in S_p$, there always exists an $f \in S$ such that (4) holds. Specifically, we may define

$$f(z) = \frac{cg(z)}{c + g(z)}$$

where $g \in S_p$ and $-c \notin g(U)$. Elementary computations show that $f \in S$ and that $f(p) = +c$.

One function of particular importance, the "Koebe Function" for the class S_p, is given by

$$K_p(z) = \frac{z}{(1 - pz)(1 - \frac{z}{p})}.$$

The logarithmic coefficients of this function are easily computed to be

$$\delta_k(K_p) = \frac{1}{2k}(p^k + \frac{1}{p^k})^2.$$

We are thus naturally led to formulate a Milin Conjecture for the class S_p:

Conjecture II. Let $g \in S_p$ and let δ_k $(k = 1, 2, \cdots)$ denote the logarithmic coefficients of g. Then, for each $N \geq 1$, we have

$$\sum_{k=1}^{N} k(N + 1 - k) \mid \delta_k \mid^2 \leq \frac{1}{4}\sum_{k=1}^{N} \frac{N + 1 - k}{k}(p^k + \frac{1}{p^k})^2$$

Equality holds if and only if $g(z) = K_p(z)$.

Remark 1. Conjectures I and II are equivalent.

Remark 2. If $N = 1$, then Conjecture II becomes $|2\delta_1|^2 \leq p + \frac{1}{p}$ for any $g \in S_p$. If g has the Taylor expansion $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ near the origin, then $b_2 = 2\delta_1$; hence, we must show that $|b_2| \leq p + \frac{1}{p}$. Now Goodman [3] conjectured that for each $g \in S_p$, the coefficient inequalities

$$| b_n | \leq \frac{1 - p^{2n}}{p^{n-1}(1 - p^2)} \tag{5}$$

should hold for each $n \geq 2$. Jenkins [4] then proved that Goodman's Conjecture for the class S_p would be true if the Bieberbach Conjecture for the class S where true. Since the Bieberbach Conjecture has been established [1], as a consequence of Milin's Inequality, the Goodman Conjecture for the class S_p is also true. In particular, (5) becomes $|b_2| \leq p + \frac{1}{p}$ when $n = 2$. Thus, Conjectures I and II are true if $N = 1$.

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