TAMKANG JOURNAL OF MATHEMATICS
Volume 39, Number 1, 63-74, Spring 2008

GENERALIZATION ON SOME THEOREMS OF L^1-CONVERGENCE OF CERTAIN TRIGONOMETRIC SERIES

ŽIVORAD TOMOVSKI

Abstract. In this paper we study L^1-convergence of the r-th derivatives of Fourier series with complex-valued coefficients. Namely new necessary-sufficient conditions for L^1-convergence of the r-th derivatives of Fourier series are given. These results generalize corresponding theorems proved by several authors (see [7], [10], [13], [19]). Applying the Wang-Telyakovskii class $(\mathbf{BV})_r^\sigma$, $\sigma > 0$, $r = 0, 1, 2, \ldots$ we generalize also the theorem proved by Garrett, Rees and Stanojević in [5]. Finally, for $\sigma = 1$ some corollaries of this theorem are given.

1. Introduction

Let $\{c_k: \ k = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of complex numbers and the partial sums of the complex trigonometric series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikt} \tag{1.1}$$

be denoted by

$$S_n(c) = S_n(c,t) = \sum_{k=-n}^{n} c_k e^{ikt}, \quad t \in (0, \pi]. \tag{1.2}$$

A sequence $\{c_k\}$ is of bounded variation of integer order $m \geq 1$, i.e. $\{c_k\} \in (BV)^m$ if $\sum_{k=-\infty}^{\infty} |\Delta^m c_k| < \infty$, where

$$\Delta^m c_k = \Delta(\Delta^{m-1} c_k) = \Delta^{m-1} c_k - \Delta^{m-1} c_{k+1}.$$

For $m = 1$, the class $(BV)^1$ is the class of complex sequences of bounded variation.

If the trigonometric series (1.1) is a Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$, for all n and the partial sums of the corresponding Fourier series are denoted by $S_n(f) = S_n(f,t) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}$.

A complex null sequence $\{c_n\}$ satisfying $\sum_{n=1}^{\infty} |\Delta(c_n - c_{-n})| \lg n < \infty$ is called weakly even (see [10]). It is obvious that if $\{c_n\}$ is an even sequence then it is weakly even.

Received August 10, 2004; revised September 10, 2007.
2000 *Mathematics Subject Classification.* 26D15, 42A20.
Key words and phrases. L^1-convergence, trigonometric series, $r-$th derivatives of Fourier series, Wang-Telyakovskii class, Sheng class, Fomin class.

In the case of complex coefficients, i.e. if $\{c_n\}$ is weakly even, the modified sums are defined as follows (see [2]):

$$G_n(c,t)=\sum_{k=0}^{n}(\Delta c_k)D_k(t)+\sum_{k=0}^{n}[\Delta(c_{-k}-c_k)](E_{-k}(t)-\frac{1}{2}),$$

where $E_k(t)=\frac{1}{2}+\sum_{j=1}^{k}e^{ijt}$ and D_k is the Dirichlet kernel. In the case of real coefficients (see [4]) Garrett and Stanojević, defined the following class C. A null sequence $\{a_n\}$ of real numbers belongs to the class C if for every $\varepsilon>0$, there exists a $\delta>0$, independent of n and such that

$$\int_0^{\delta}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(t)\right|dt<\varepsilon \quad \text{for all } n.$$

Let

$$S_n(a)=S_n(a,t)=\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos kt,\quad t\in(0,\pi], \tag{1.3}$$

where $\{a_n\}$ is a real null sequence of bounded variation of integer order $m\ge 1$, i.e. $\{a_n\}\in(\mathbf{BV})^m$. Then $\lim_{n\to\infty}S_n(a,t)=f(t)$ exists in $(0,\pi]$ (see [5]).

Garrett, Rees and Stanojević (see [5]) have given necessary and sufficient conditions for series (1.3) to be Fourier series of some $f\in L^1(0,\pi)$. Namely they have proved the following theorem.

Theorem A. *Let $\{a_n\}\in(\mathbf{BV})^m$, $m=1,2,3,\ldots$ and $a_n\lg n=o(1)$, $n\to\infty$. Then*

$$\|S_n-f\|=o(1),\quad n\to\infty \quad \text{if and only if} \quad \{a_n\}\in C.$$

The difference of noninteger order $\sigma\ge 0$ of the sequence $\{a_n\}_{n=0}^{\infty}$ is defined as follows:

$$\Delta^{\sigma}a_n=\sum_{m=0}^{\infty}\binom{m-\sigma-1}{m}a_{n+m}\quad (n=0,1,2,\ldots)$$

where

$$\binom{m+\alpha}{m}=\frac{(1+\alpha)\cdots(m+\alpha)}{m!}.$$

Wang and Telyakovskii (see [20]) have considered the following class of real sequences $\{a_n\}$.

Namely, a null-sequence $\{a_n\}$ belongs to the class $(\mathbf{BV})_r^{\sigma}$, $r=0,1,2,\ldots,\sigma\ge 0$ if $\sum_{k=1}^{\infty}k^r|\Delta^{\sigma}a_k|<\infty$.

Theorem B.([20]) *Let $\rho\ge 0$, $\sigma\ge 0$. Then for all $\gamma>\sigma$ the following embedding relation holds,*

$$(\mathbf{BV})_{\rho}^{\sigma}\subset(\mathbf{BV})_{\rho}^{\gamma}.$$

In the same paper, Wang and Telyakovskii by considering the complex form of trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{inx}, \quad x \in (0,\pi]$$

have proved the following theorem.

Theorem C. *If*

$$\{a_k\} \in (\mathbf{BV})_r^{\sigma}, \quad r = 0,1,2,\ldots, \quad \sigma \geq 0$$

then cosine series (1.3) *and sine series* $\sum_{n=1}^{\infty} a_n \sin nx$ *have derivatives of r-th order on* $(0,\pi]$.

The Wang-Telyakovskii class $(\mathbf{BV})_r^{\sigma}$, $r = 0,1,2,\ldots$, $\sigma \geq 0$, motivated us to consider a further class $(BV)_r^m$, $r = 0,1,2,\ldots$, $m = 1,2,3,\ldots$ (see [18]) of complex null-sequences $\{c_n\}$ such that

$$\sum_{k=-\infty}^{\infty} |k|^r |\Delta^m c_k| < \infty.$$

For $r = 0$, we have $(BV)_r^m = (BV)^m$.

On the other hand in [14], we have defined the extension C_r, $r = 1,2,3,\ldots$ of the Garret-Stanojević class C as follows:

A null real sequence $\{a_n\}$ belongs to the class C_r, $r = 1,2,3,\ldots$ if for every $\varepsilon > 0$, there exists $\delta > 0$, independent of n and such that

$$\int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta a_k D_k^{(r)}(x) \right| dx < \varepsilon, \quad \text{for all} \quad n.$$

V. B. Stanojević in [9] defined the class C^* of all weakly even complex sequences such that for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, independent of n, and

$$\int_{|t|\leq\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k(t) \right| dt < \varepsilon, \quad \text{for all} \ n.$$

In this paper, we shall consider complex null-sequences $\{c_n\}$ such that

$$\sum_{n=1}^{\infty} |\Delta(c_n - c_{-n})| \, n^r \lg n < \infty. \tag{1.4}$$

Let C_r^* denote the class of all complex null sequences $\{c_n\}$ such that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, independent of n and such that

$$\int_{|t|\leq\delta} \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right| dt < \varepsilon, \quad \text{for all} \quad n.$$

Let

$$\lim_{n\to\infty} S_n^{(r)}(f,t) = f_r(t), \quad r = 1,2,3,\ldots$$

If $f_r \in L^1$, then it is denoted by $f^{(r)}(t)$.

We note that the class $(BV)_r^m$, $r = 1,2,\ldots$, $m = 1,2,3,\ldots$, for $m = 1$ is the class $(BV)_r$, defined in [13].

In [13], we have proved the following theorem.

Theorem D. *Let* $\{c_n\}$ *is a complex sequence such that* (1.4) *holds. If*

$$\{c_n\} \in (BV)_r \cap C_r^*,$$

then

$$\|S_n^{(r)} - f^{(r)}\| = o(1), \quad n \to \infty$$

if and only if

$$n^r |c_n| \lg n = o(1), \quad n \to \infty.$$

In this paper, we shall extend the Theorem D, by considering the class $(BV)_r^m$, $r = 1,2,\ldots$, $m = 1,2,3,\ldots$ instead of $(BV)_r$.

In addition we shall give the extension of the Theorem A, by considering the Wang-Telyakovskii class $(\mathbf{BV})_r^\sigma$, $r = 0,1,2,\ldots$, $\sigma > 0$ instead of $(\mathbf{BV})^m$, $m = 1,2,3,\ldots$.

2. Lemmas

For the proofs of the our main results, we need the following Lemmas:

Lemma 1.([7]) *For each* $r = 0,1,2,\ldots$ *the following inequality holds*

$$\|E_{-n}^{(r)}(t)\| = O(n^r \lg n).$$

Lemma 2.([7]) *For each nonnegative integer n, there holds*

$$\|c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)\| = o(1), \quad n \to \infty$$

if and only if

$$n^r |c_n| \lg n = o(1), \quad n \to \infty.$$

We note that this lemma for $r = 0$, was proved by W. Bray and Č. V. Stanojević in [1].

Lemma 3. *For all* $p \geq 1$, $r = 0,1,2,\ldots$ *and* $\delta > 0$ *the following estimate holds*

$$\int_{|t|>\delta} \left| \frac{d^r}{dt^r} \left(\frac{e^{it}}{e^{it} - 1} \right)^p \right| dt = O_{p,r,\delta}(1), \quad t \in (0,\pi],$$

where $O_{p,r,\delta}$ *depends on* p, r *and* δ.

Proof. See the proof of Lemma 1 in [18].

Lemma 4. *Let* $\{c_k\} \in (BV)_r^m$, $m = 1,2,3,\ldots$, $r = 1,2,3,\ldots$ *and* $n^r|c_n|\lg n = o(1)$, $n \to \infty$. *Then for* $\delta > 0$ *the following limit holds*

$$\int_{|t|>\delta} \left| \sum_{k=n+1}^{\infty} \Delta c_k D_k^{(r)}(t) \right| dt = o(1), \quad n \to \infty,$$

where $t \in (0,\pi]$.

Proof. It is easy to prove that

$$\sum_{k=n+1}^{\infty} (\Delta c_k) E_k(t) = \sum_{k=n+1}^{\infty} c_k e^{ikt} + c_{n+1} E_n(t).$$

Now we consider the identity, obtained by V. B. Stanojević in [8].

$$\begin{aligned} \omega^m \sum_{j=M+m}^{N+m} c_j e^{ijt} = & \sum_{j=M}^{N} (\Delta^m c_j)\, e^{ijt} \\ & - \sum_{k=0}^{m-1} \omega^k [(\Delta^{m-k-1} c_{M+k}) e^{i(M+k)t} \\ & -(\Delta^{m-k-1} c_{N+k+1}) e^{i(N+k+1)t}], \end{aligned}$$

where $\omega = 1 - e^{-it}$.

Setting $M = n$ and letting $N \to \infty$, for $t \neq 0$, we obtain:

$$\begin{aligned} \omega^m \sum_{j=m+n}^{\infty} c_j e^{ijt} = & \sum_{j=n}^{\infty} (\Delta^m c_j)\, e^{ijt} \\ & - \sum_{k=0}^{m-1} \omega^k \left[(\Delta^{m-k-1} c_{n+k})\, e^{i(n+k)t} \right], \end{aligned}$$

i.e.

$$\begin{aligned} \sum_{j=n+1}^{\infty} c_j e^{ijt} = & \; \omega^{-m} \sum_{j=n}^{\infty} (\Delta^m c_j) e^{ijt} \\ & -\omega^{-m} \sum_{k=0}^{m-1} \omega^k \left[(\Delta^{m-k-1} c_{n+k}) e^{i(n+k)t} \right] + \sum_{j=n+1}^{m+n-1} c_j e^{ijt}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{k=n+1}^{\infty} (\Delta c_k) E_k(t) = & \left(\frac{e^{it}}{e^{it}-1} \right)^m \sum_{j=n}^{\infty} (\Delta^m c_j)\, e^{ijt} \\ & - \left(\frac{e^{it}}{e^{it}-1} \right)^m \sum_{k=0}^{m-1} \sum_{q=0}^{k} (-1)^q \binom{k}{q} e^{it(n+k-q)} (\Delta^{m-k-1} c_{n+k}) \\ & + \sum_{j=n+1}^{m+n-1} c_j e^{ijt} + c_{n+1} E_n(t). \end{aligned}$$

Applying the inequality (see [17])

$$|E_j^{(r)}(t)| \le \frac{4\pi j^r}{|t|}, \quad 0 < |t| \le \pi \tag{2.1}$$

we obtain that the series $\sum_{j=n+1}^{\infty}(\Delta c_j)E_j^{(r)}(t)$ is uniformly convergent on any compact subset of $(0,\pi]$. Also, by $\{c_n\} \in (BV)_r^m$, we obtain that the series $\sum_{j=n+1}^{\infty} j^r (\Delta^m c_j)\, e^{ijt},\ m = 1,2,3,\ldots$ is uniformly convergent on any compact subset of $(0,\pi]$.

Hence,

$$\begin{aligned}\sum_{k=n+1}^{\infty}(\Delta c_k)E_k^{(r)}(t) &= \sum_{\nu=0}^{r}\binom{r}{\nu}\frac{d^\nu}{dt^\nu}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m i^{r-\nu}\sum_{j=n}^{\infty} j^{r-\nu}(\Delta^m c_j)e^{ijt}\\ &\quad-\sum_{\nu=0}^{r}\binom{r}{\nu}\frac{d^\nu}{dt^\nu}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\sum_{k=0}^{m-1}\sum_{q=0}^{k}(-1)^q\binom{k}{q}(n+k-q)^{r-\nu}i^{r-\nu}\\ &\quad\times e^{it(n+k-q)}(\Delta^{m-k-1}c_{n+k})+\sum_{j=n+1}^{m+n-1} i^r c_j j^r e^{ijt}\\ &\quad+c_{n+1}E_n^{(r)}(t).\end{aligned}$$

Then,

$$\begin{aligned}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_k^{(r)}(t)\right| &\le \sum_{\nu=0}^{r}\binom{r}{\nu}\left|\frac{d^\nu}{dt^\nu}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\right|\sum_{j=n}^{\infty} j^r|\Delta^m c_j|\\ &\quad+\sum_{\nu=0}^{r}\binom{r}{\nu}\left|\frac{d^\nu}{dt^\nu}\Big(\frac{e^{it}}{e^{it}-1}\Big)^m\right|\sum_{k=0}^{m-1}\sum_{q=0}^{k}\binom{k}{q}(n+k-q)^r|\Delta^{m-k-1}c_{n+k}|\\ &\quad+\sum_{j=n+1}^{m+n-1} j^r|c_j|+|c_{n+1}|\,|E_n^{(r)}(t)|.\end{aligned}$$

Applying Lemma 3 and (2.1), we obtain

$$\begin{aligned}\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_k^{(r)}(t)\right|dt &\le O_{m,r,\delta}(1)\sum_{j=n+1}^{\infty} j^r|\Delta^m c_j|\\ &\quad+O_{m,r,\delta}(1)\left(\sum_{k=0}^{m-1}\sum_{q=0}^{k}\binom{k}{q}(n+k-q)^r|\Delta^{m-k-1}c_{n+k}|\right)\\ &\quad+O_\delta\left(\sum_{j=n+1}^{m+n-1} j^r|c_j|\right)+O_\delta(|c_{n+1}|n^r).\end{aligned}\tag{2.2}$$

Since

$$\sum_{k=1}^{m-1}\sum_{q=0}^{k}\binom{k}{q}(n+k-q)^r|\Delta^{m-k-1}c_{n+k}| \le \sum_{k=0}^{m-1}2^k(n+k)^r|\Delta^{m-k-1}c_{n+k}|$$

$$\le 2^{m-1}\sum_{j=n}^{m+n-1} j^r|\Delta^{m+n-j-1}c_j|$$
$$\le 2^{m-1}\sum_{j=n}^{m+n-1}\sum_{k=0}^{m+n-j-1}\binom{m+n-j-1}{k} j^r|c_{j+k}|$$

and $n^r|c_n|\lg n = o(1), n\to\infty$, the second sum on the right-hand side of the inequality (2.2) is finite sum of $o(1)$-terms as $n\to\infty$.

Hence,

$$\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_k^{(r)}(t)\right|dt \le O_{m,r,\delta}(1)\sum_{j=n+1}^{\infty} j^r|\Delta^m c_j| = o(1),\quad n\to\infty. \tag{2.3}$$

Since $E_{-k}^{(r)}(t)=E_k^{(r)}(-t)$, by (2.3), we obtain

$$\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_{-k}^{(r)}(t)\right|dt = o(1),\quad n\to\infty.$$

Using the equality

$$D_n(t)=\frac{1}{2}(E_n(t)+E_{-n}(t))$$

we obtain

$$\begin{aligned}\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)D_k^{(r)}(t)\right|dt &\le \frac{1}{2}\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_k^{(r)}(t)\right|dt\\ &\quad+\frac{1}{2}\int_{|t|>\delta}\left|\sum_{k=n+1}^{\infty}(\Delta c_k)E_{-k}^{(r)}(t)\right|dt\\ &= o(1),\quad n\to\infty.\end{aligned}$$

3. Main result

Theorem 1. *Let* $\{c_n\}$ *be a sequence of complex numbers such that* (1.4) *holds. If*

$$\{c_n\}\in(BV)_r^m\cap C_r^*,\quad m=1,2,3,\ldots,\quad r=1,2,3,\ldots$$

then

$$\|S_n^{(r)}-f^{(r)}\| = o(1),\quad n\to\infty$$

if and only if

$$n^r|c_n|\lg n = o(1),\quad n\to\infty.$$

Proof. Sufficiency: Assume that

$$\{c_n\}\in(BV)_r^m\cap C_r^*,\quad n^r|c_n|\lg n = o(1),\quad n\to\infty$$

and (1.4) holds. Then (see [18])

$$\lim_{n\to\infty} S_n^{(r)}(c,t) = f_r(t)\,. \tag{$*$}$$

Let

$$\begin{aligned} G_{n,r}(c,t) &= S_n^{(r)}(c,t) - \left(c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)\right) \\ &= \sum_{k=1}^{n} (\Delta c_k) D_k^{(r)}(t) + \sum_{k=1}^{n} \Delta(c_{-k} - c_k)\left(E_{-k}^{(r)}(t)\right). \end{aligned}$$

For $t \neq 0$ it follows from $(*)$ that

$$f_r(t) - G_{n,r}(c,t) = \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) + \sum_{k=n+1}^{\infty} \Delta(c_k - c_{-k})\left(E_{-k}^{(r)}(t)\right).$$

From Lemma 1, (1.4) and Lemma 4, we obtain

$$\begin{aligned} \| f_r(t) - G_{n,r}(c,t)\| &\le \left\| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right\| + \alpha \sum_{k=n+1}^{\infty} |\Delta(c_k - c_{-k})| k^r \lg k \\ &= \left(\int_{|t|\le\delta} + \int_{|t|>\delta} \right) \left| \sum_{k=n+1}^{\infty} (\Delta c_k) D_k^{(r)}(t) \right| dt + o(1) = o(1)\,, \quad n\to\infty, \end{aligned}$$

i.e.

$$\| f_r(t) - G_{n,r}(c,t)\| = o(1)\,, \quad n\to\infty.$$

Applying Lemma 2, we obtain

$$\|G_{n,r}(c,t) - S_n^{(r)}(c,t)\| = \|c_n E_n^{(r)}(t) + c_{-n} E_{-n}^{(r)}(t)\| = o(1)\,, \quad n\to\infty.$$

Hence,

$$\|S_n^{(r)}(c,t) - f_r(t)\| \le \| f_r(t) - G_{n,r}(c,t)\| + \|G_{n,r}(c,t) - S_n^{(r)}(c,t)\| = o(1)\,, \quad n\to\infty,$$

i.e. $f_r \in L^1$. Thus

$$\lim_{n\to\infty} S_n^{(r)}(c,t) = f^{(r)}(t)\,.$$

Necessity: Let $\{c_n\} \in (BV)_r^m \cap C_r^*$, (1.4) holds and $\|S_n^{(r)}(c,t) - f^{(r)}(t)\| = o(1)\,,\ n\to\infty$.
Applying Lemma 2, it suffices to show that

$$\| \hat{f}(n) E_n^{(r)}(t) + \hat{f}(-n) E_{-n}^{(r)}(t)\| = o(1)\,, \quad n\to\infty.$$

Indeed,

$$\begin{aligned} \| \hat{f}(n) E_n^{(r)}(t) + \hat{f}(-n) E_{-n}^{(r)}(t)\| &= \|G_{n,r}(c,t) - S_n^{(r)}(c,t)\| \\ &\le \|G_{n,r}(c,t) - f_r(t)\| + \| f_r(t) - S_n^{(r)}(c,t)\| \\ &= o(1)\,, \quad n\to\infty, \end{aligned}$$

i.e.

$$n^r \hat{f}(n) \lg n = o(1)\,, \quad n\to\infty.$$

Theorem 2. *Let* $\{a_n\} \in (\mathbf{BV})_r^{\sigma} \cap C_r$, *where* $\sigma > 0$ *and* $r = 0, 1, 2, \ldots$ *Then the* r*-th derivative of the series* (1.3) *is a Fourier series of some* $f^{(r)} \in L^1(0, \pi]$ *and*

$$\| S_n^{(r)} - f^{(r)} \| = o(1), \quad n \to \infty$$

if and only if

$$n^r a_n \lg n = 0(1), \quad n \to \infty.$$

Proof. Let m be the least integer such that $m \geq \sigma$. Then by Theorem B, we obtain $\{a_n\} \in (\mathbf{BV})_r^m$ and by Theorem C, the point-wise limit $f^{(r)}$ of the r-th derivative of the sum $S_n(a)$ exists in $(0, \pi]$. Applying the same technique for series (1.3), as in the proof of Theorem 1, the proof of this theorem is obvious.

4. Some corollaries for $\sigma = 1$

Firstly, we shall define some known classes of real sequences introduced in [12], [14], [15], [16], [17], [19].

A null-sequence $\{a_k\}$ belongs to the class S_r, $r = 0, 1, 2, 3, \ldots$, (see [15], [17]) if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=0}^{\infty} k^r A_k < \infty$ and $|\Delta a_k| \leq A_k$, for all k.

When $r = 0$, we obtain the Sidon-Telyakovskii class S (see [11]).

A null sequence $\{a_k\}$ belongs to the class S_{qr}, $q > 1$, $r = 0, 1, 2, \ldots$ (see [14]) if there exists a monotonically decreasing sequence $\{A_k\}$ such that

$$\sum_{k=1}^{\infty} k^r A_k < \infty \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1).$$

It is clear that $S_r \subset S_{qr}$.

On the other hand, in [16] we defined an equivalent form of the Sheng's class $S'_{q\alpha r}$, $q > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$ (see [7]) as follows: a null sequence $\{a_k\}$ belongs to the class $S_{q\alpha r}$, $q > 1$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$ if there exists a monotonically decreasing sequence $\{A_k\}$ such that $\sum_{k=1}^{\infty} k^{\alpha} A_k < \infty$ and

$$\frac{1}{n^{q(\alpha - r) + 1}} \sum_{k=1}^{n} \frac{|\Delta a_k|^q}{A_k^q} = O(1).$$

The following embedding relation holds (see [16])

$$S_{q\alpha r} \subset (BV)_r \cap C_r, \quad 1 < q \leq 2, \quad \alpha \geq 0, r \in \{0, 1, 2, \ldots, [\alpha]\}.$$

Corollary 4.1. *Let* $\{a_n\} \in S_{q\alpha r}$, $1 < q \leq 2$, $\alpha \geq 0$, $r \in \{0, 1, 2, \ldots, [\alpha]\}$. *Then the* r*-th derivative of the series* (1.3) *is a Fourier series of some* $f^{(r)} \in L^1(0, \pi)$ *and*

$$\| S_n^{(r)} - f^{(r)} \| = o(1), \quad n \to \infty$$

if and only if

$$n^r a_n \lg n = o(1), \quad n \to \infty.$$

We note that for $\alpha = r$ we obtain analogical results for the classes S_{qr} and S_r, and for $r = 0$ for the Sidon-Telyakovskii class S.

Denote by I_m the diadic interval $[2^{m-1}, 2^m)$, for $m \geq 1$.

A null sequence $\{a_n\}$ belongs to the class $F_{qr},\ q > 1,\ r = 0, 1, 2, \ldots$ if

$$\sum_{m=1}^{\infty} 2^{m(1+r)} \left(\frac{1}{2^m} \sum_{k \in I_m} |\Delta a_k|^q \right)^{1/q} < \infty.$$

It is obvious that for $r = 0$, we obtain the Fomin's class F_q. $q > 1$ (see [3]).

But in [12] we verified the embedding relation

$$F_{qr} \subset (BV)_r \cap C_r, 1 < q \leq 2, r = 1, 2, \ldots.$$

Corollary 4.2. *Let*

$$\{a_n\} \in F_{qr}, \quad 1 < q \leq 2, \quad r = 0, 1, 2, \ldots.$$

Then the r-th derivative of the series (1.3) *is a Fourier series of some $f^{(r)} \in L^1(0, \pi)$ and*

$$\| S_n^{(r)} - f^{(r)} \| = o(1), \quad n \to \infty$$

if and only if

$$n^r a_n \lg n = o(1), \quad n \to \infty.$$

References

[] W. O. Bray and Č. V. Stanojević, *Tauberian L^1-convergence classes of Fourier series*, Trans. Amer. Math. Soc. **275** (1983), 59–69.

[] W. O. Bray and Č. V. Stanojević, *Tauberian L^1-convergence classes of Fourier series* II, Math. Ann. **269** (1984), 469–486.

[] G. A. Fomin, *A class of trigonometric series*, Mat. Zametki, **23** (1978), 117–124, (Russian).

[] J. W. Garrett and Č. V. Stanojević, *On L^1-convergence of certain cosine sums*, Proc. Amer. Math. Soc. **54** (1976), 101–105.

[] J. W. Garrett, C. S. Rees and Č. V. Stanojević, *L^1-convergence of Fourier series with coefficients of bounded variation*, Proc. Amer. Math. Soc. **80** (1980), 423–430.

[] C. S. Rees, Č. V. Stanojević, *Necessary and sufficient conditions for integrability of certain cosine sums*, J. Math. Anal. Appl. **43** (1973), 579–586.

[] S. Sheng, *The extension of the theorems of Č. V. Stanojević and V. B. Stanojević*, Proc. Amer. Math. Soc. **110** (1990), 895–904.

[] V. B. Stanojević, *On a theorem of P. L. Uljanov*, Proc. Amer. Math. Soc. **90**(1984), 370–372.

[] V. B. Stanojević, *Convergence and integrability of trigonometric series with coefficients of bounded variation of order (m, p)*, Proc. Amer. Math. Soc. **114** (1992), 711–718.

[] Č. V. Stanojevic, V. B. Stanojević, *Generalization of the Sidon-Telyakovskii theorem*, Proc. Amer. Math. Soc. **101** (1987), 679–684.

[] S. A. Telyakovskii, *On a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki **14** (1973), 742–748, (Russian).

[] Ž. Tomovski, *On a Bojanić-Stanojević Type inequality and its Applications*, Journal of Ineq. in Pure and Appl. Math. **1** (2), Article 13, (2000). URL: http://jipam.vu.edu.au

[] Ž. Tomovski, *Necessary and sufficient condition for L^1-convergence of complex trigonometric series*, VESTNIK, Russian University of People's Friendship, Moscow, **7**(2000), 139–145.

[] Ž. Tomovski, *An extension of the Garrett-Stanojević class*, (preprint) RGMIA Research report collection, **3** (4), Article 3, (2000). URL: http: //melba.vu.edu.au/~rgmia/v3n4.html

[] Ž. Tomovski, *An extension of the Sidon-Fomin type inequality and its Applications*, Math. Ineq. & Applications **4** (2) (2001), 231–238.

[] Ž. Tomovski, *Some classes of L^1-convergence of Fourier series*, Journal of Comput. Anal. & Appl. **4** (2002), 79–89.

[] Ž. Tomovski, *Some results on L^1-approximation of the r-th derivative of Fourier series*, Journal of Ineq. in Pure and Appl. Math. 3(1), Article 10, (2002). URL: http: //jipam.vu.edu.au

[] Ž. Tomovski, *Some classes of L^p $(0 < p < 1)$ convergence of trigonometric series*, Tamkang J. Math. **33** (2002), 275–283.

[] Ž. Tomovski, *Convergence and integrability on some classes of trigonometric series*, Dissert. Math. 420 (2003), Warszawa.

[] K. Wang and S. A. Telyakovskii, *Differential properties of sums of a class of trigonometric series*, (Russian), Moscow Univ. Math. Bull. **54** (1999), N∘.1, 26–30.

Faculty of Mathematical and Natural Sciences, Department of Mathematics, P.O. BOX 162, 1000 Skopje, MACEDONIA

E-mail: tomovski@iunona.pmf.ukim.edu.mk; zivoradt@yahoo.com