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A NOTE ON RECURSIVE ESTIMATOR OF THE DENSITY FUNCTION WHICH IS NOT NECESSARY CONTINUOUS

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Summary. Consider a sequence $X_1, X_2, \ldots, X_n$ of independent, identically distributed random variables with nukown density function f, which with its first derivative are not necessarily continuous, and let

$$f_n^*(x) = (n^2 h_n)^{-1/2} \sum_{j=1}^{n} h_j^{-1/2} K(\frac{x - X_j}{h_j})$$

be the recursive kernel estimator of f. It will be shown, under certain additional regularity conditions on K and $|h(n)|$, that, $MISE\ [f_n^*(x)] = 0\ (n^{-4/5})$ if f and f' are continuous, where as $MISE\ [f_{n_1}^*(x)] = 0\ (n^{-3/4})$ if f is continuous and f' is not and $MISE\ [f_{n_2}^*(x)] = 0\ (n^{-1/2})$ if f and f' are not continuous.

Introduction:

Consider a sequence $X_1, X_2, \ldots, X_n$ of n independent, identically distributed random variables (i.i.d.) with unknown density function (d.f.) $f(x)$. A class of density estimators, known as Kernel estimators, which has been widely studied by Parzen [1] and Rosenblatt [2] may be defined as follows:

$$\hat{f}_n(x) = (nh)^{-1} \sum_{j=1}^{n} K(\frac{x - X_j}{h}),$$

Parzen has shown that if the Kernel function $K(.)$ satisfies certain conditions, including $\int_{-\infty}^{\infty} K(y)dy = 1$ and $h = h(n)$ is a bounded and nonincreasing function of n such that $h(n) = 0(n^{-1})$, then $\hat{f}(x)$ is a mean squared error consistnet estimator of $f(x)$ at all continuity points x of f.

E. J. Wegman and H. I. Davies [3] considered a modified estimator of the form

$$f_n^*(x) = (n^2 h_n)^{-1/2} \sum_{j=1}^{n} (h_j)^{-1/2} K(\frac{x - X_j}{h_j}). \qquad (1)$$

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They established laws of the iterated logarithm for density estimators as well as the asymptotic distribution results using the almost sure invariance principle.

In this article we shall discuss classes of estimators of the form $f_n^*(x)$ for an unknown d.f. $f(x)$ which with its first derivative are not necessarily continuous. As a measure of improvement, many authors use the mean integrated square error given by

$$MISE\ [f_n^*(x)] \ = E \int_{-\infty}^{\infty} [f_n^*(x) - f(x)]^2 dx \tag{2}$$

Let

$$M_1 ISE\ [f_n^*(x)] \ = E \int_{-\infty}^{\infty} [f_n^*(x) - B(f(x))]^2 \ dx; \tag{3}$$

where

$$B(f(x)) \ = E\ [f_n^*(x)] - f(x).$$

Hence

$$MISE\ [f_n^*(x)] \ = M_1 ISE\ [f_n^*(x)] + (1-B)^2 \ || \ f(x) \ ||^2 .$$

Consider a recursive Kernel estimator of the form (1) of f based on the random sample. Assuming $K(.)$ is a some suitable Kernel function satisfying the following conditions:

$$(A) \begin{cases} \sup_x \ | \ K(x) \ | < \infty, & \lim_{x \to \pm\infty} \ | \ xK(x) \ | = \ 0, \\ \int_{-\infty}^{\infty} \ | \ K(x) \ | \ dx < \infty \ and & \int_{-\infty}^{\infty} x^2 K(x) dx \ < \ \infty, \end{cases}$$

and $h(n)$ is a sequence of nonincreasing positive constants converging to zero as n tends to ∞.

As symbols, we shall use $f_{n_0}^*(x)$ as an estimator for f when f and f' are continuous, if f is continuous but f' is not, then we take $f_{n_1}^*(x)$ as an estimator for f and $f_{n_2}^*(x)$ as an estimator for f when f and f' are not continuous.

Main results:

Theorem 1. *Assum that $K(.)$ is a Borel function satisfies conditions (A) and $nh_n \to \infty$ as $n \to \infty$, if $f(x)$, $f'(x)$ are continous and f'' is square integrable, then*

$$MISE\ [f_{n_0}^*(x)] = C_1\{(nh_n)^{-1} \ || \ K(t) \ ||^2 + 0((nh_n)^{-1}) + [1 - (n^2 h_n)^{-1/2} r_n]^2$$
$$\times \ || \ f(x) \ ||^2 + \frac{1}{2}(nh_n)^{-1} [\sum_{j=1}^{n} h_j^5 \ || \ tK^{1/2}(t) \ ||^4 \ || \ f''(x) \ ||^2 + 0(h_j^5)]\};$$

where $0 < c_1 < 1$, $||g(t)||^2 = \int g^2(t)dt$
and

$$r_n \ = \ \sum_{j=1}^{n} h_j^{1/2}.$$

If $(n^2 h_n)^{-1/2} r_n \to B < \infty$ *and* $(nh_n)^{-1} \sum_{j=1}^{n} h_j^5 \to 0$ *as* $n \to \infty$, *then*

$$MISE\ [f_n^*(x)] \to (1-B)^2\ \ ||\ f(x)\ ||^2 \text{ as } n \to \infty,$$

and hence

$$M_1ISE\ [f_n^*(x)] \to 0 \text{ as } n \to \infty.$$

In particular if $h_j = 0\ (n^{-r}),\ j = 1,2,\ldots,n;\ r > 0,$ *then*

$$M_1ISE\ [f_n^*(x)]\ =\ 0(n^{r-1})\ +\ 0(n^{-4r}),$$

and hence the asymptotic optimum value r_0 *of* r *is* $\frac{1}{5}$ *and then we get minimum* $M_1ISE\ [f_{n_0}^*(x)]\ =\ 0(n^{-4/5})$.

Theorem 2. *Suppose* $K(.)$ *and* h_n *satisfy conditions of theorem 1.*

If $f(x)$ *is continuous,* f' *has finite points of discontinuity* $(b_1, b_2, \ldots, b_\ell)$ *and* f'' *is square integrable, then*

$$M_1ISE\ [f_{n_1}^*(x)]\ =\ C_2\{M_1ISE\ [f_{n_0}^*(x)] + \frac{4\Delta}{n}(\sum_{j=1}^{n} h_j^3) \int_{-\infty}^{0} [\int_{-\infty}^{y} (y-t)K(t)dt]^2\ dy$$

$$+\, O(\frac{1}{n}\sum_{j=1}^{n} h_j^3)\},$$

where $0 < c_2 < 1,\ \Delta = \sum_{j=1}^{\ell}\ \Delta_j^2,$

$$\Delta_j\ =\ f'(b_j^-) - f'(b_j^+),\ j = 1,2,\ldots,n,$$

If $\frac{1}{n}\sum_{j=1}^{n} h_j^3\ \to 0$ *as* $n \to \infty$, *then as* $n \to \infty$, *the estimation* $f_{n_1}^*(x)$ *is consistent in* M_1ISE *sense, specially if* $h_j = 0(n^{-r}),\ j = 1,2,\ldots,n,\ r > 0,$ *then* $M_1ISE\ [f_{n_1}^*(x)] = 0(n^{r-1}) + 0(n^{-3r})$ *and we get the asymptotic optimum value* r_0 *of* r *is* $\frac{1}{4}$, *and hence minimum* $M_1ISE[f_{n_1}^*(x)] = 0(n^{-3/4})$.

Theorem 3. *Let* $K(.)$ *and* h_n *satisfy conditions of theorem 1.*

If f *has* $k(k \geq 0)$ *points of discontinuity,* $-\infty = a_0 < a_1 < a_2 < \ldots < a_k < a_{k+1} = \infty$, *and for each* $i = 1,\ldots,k+1, f'$ *has* $\ell_i(\ell_i \geq 0)$ *points of discontinuity.*

$$a_{i-1} = b_{i_0} < b_{i_1} < b_{i_2} < \ldots < b_{i_{\ell_i}} < b_{i_{\ell_i+1}}\ \ =\ a_i, \text{ also}$$

$$\int_{-\infty}^{\infty} |\ f'(x)\ |\ dx < \infty, \qquad \int_{-\infty}^{\infty} |\ f''(x)\ |^2\ dx < \infty$$

then

$$M_1ISE\ [f_n^*(x)]\ <\ c_2\ M_1ISE\ [f_{n_1}^*(x)] + \frac{4\delta}{n}(\sum_{j=1}^{n} h_j) \int_{0}^{\infty} |\int_{y}^{\infty} K(x)dx\ |^2\ \ dy$$

$$+\, 0(\frac{1}{n}\sum_{j=1}^{n} h_j),$$

where $\delta = \sum_{i=1}^{k} \delta_i^2, \qquad \delta_i = f(a_i^-) - f(a_i^+),$

$$\Delta = \sum_{i=1}^{k+1} \sum_{j=1}^{\ell_i+1} \Delta_{ij}^2 \quad \text{and} \quad \Delta_{ij} = f'(b_{ij}^-) - f'(b_{ij}^+),$$

$$i = 1, \cdots, k+1 \quad \text{and} \quad j = 1, \cdots, \ell_i + 1.$$

If $\frac{1}{n}\sum_{j=1}^{n} h_j \to 0$ *as* $n \to \infty$, *then the estimation* $f_{n_2}^*(x)$ *is consistent in* M_1ISE *sense. In particular if* $h_j = 0(n^{-r})$, $j = 1, 2, \cdots, n$, $r > 0$, *then* $M_1ISE\ [f_{n_2}^*(x)] = 0(n^{r-1}) + 0(n^{-r})$ *and also we get* $r_0 = \frac{1}{2}$ *and minimum* $M_1ISE\ [f_{n_2}^*(x)] = 0(n^{-1/2})$.

Proof: It is sufficient to prove theorem 3 which implies theorem 1 and 2.
We can see that

$$\begin{aligned}
MISE\ [f_{n_2}^*(x)] =& E\int_{-\infty}^{\infty} [f_n^*(x) - f(x)]^2\, dx \\
=& E\int_{-\infty}^{\infty} [(n^2h_n)^{-1/2}\sum_{j=1}^{n} h_j^{-1/2}K(\frac{x-X_j}{h_j}) - f(x)]^2\, dx \\
=& \int_{-\infty}^{\infty} [(h^2h_n)^{-1}\sum_{j=1}^{n} h_j^{-1}EK^2(\frac{x-X_j}{h_j}) \\
&+ 2(n^2h_n)^{-1}\sum_{1=i<j}^{n} (h_ih_j)^{-1/2}EK(\frac{x-X_i}{h_i})K(\frac{x-X_j}{h_j}) \\
&- 2f(x)(n^2h_n)^{-1/2}\sum_{j=1}^{n}(h_j)^{-1/2}EK(\frac{x-X_j}{h_j}) + f^2(x)]dx. \qquad (3)
\end{aligned}$$

First

$$\begin{aligned}
\int_{-\infty}^{\infty} h_j^{-1}EK^2(\frac{x-X_j}{h_j})dx =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_j^{-1}K^2(\frac{x-y}{h_j})f(y)dydx \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K^2(t)f(x-th_j)dtdx \\
=& \int_{-\infty}^{\infty} K^2(t)dt + 0(1). \qquad (4)
\end{aligned}$$

Second

$$\int_{-\infty}^{\infty} h_j^{-1}f(x)EK(\frac{x-X_j}{h_j})dx = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} K(t)f(x-th_j)dtdx. \qquad (5)$$

At last

$$\sum_{1=i<j}^{n} \int_{-\infty}^{\infty} (h_i h_j)^{-1/2} EK(\frac{x-X_i}{h_i}) EK(\frac{x-X_j}{h_j}) dx$$

$$= \sum_{1=i<j}^{n} (h_i h_j)^{1/2} \{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t)K(s)[f(x-th_i)-f(x)]$$

$$\times [f(x-sh_j)-f(x)]dt\ ds\ dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t)K(s)f(x)[f(x-th_i)$$

$$+ f(x-sh_j)]dt\ ds\ dx - \int_{-\infty}^{\infty} f^2(x)dx \}. \tag{6}$$

It is easy to see that the second term in the last equation is equal to

$$r_n \sum_{j=1}^{n} h_j^{1/2} \int_{-\infty}^{\infty} f(x) \int_{-\infty}^{\infty} K(t) f(x-th_j) dt\ dx;\ r_j = \sum_{j=1}^{n} h_j^{1/2}.$$

from (3), (4), (5) and (6) we get

$$MISE\ [f_{n_2}^*(x)] = (n\ h_n)^{-1} \int_{-\infty}^{\infty} K^2(t)dt$$

$$+ 2(n^2 h_n)^{-1} \sum_{1=i<j}^{n} (h_i h_j)^{1/2} \int_{-\infty}^{\infty} J(x,h_i)J(x,h_j)dx$$

$$+ 2r_n(n^2 h_n)^{-1} \sum_{j=1}^{n} h_j^{1/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(t)f(x)f(x-th_j)dt\ dx$$

$$- (n^2 h_n)^{-1} r_n^2 \int_{-\infty}^{\infty} f^2(x)dx - 2(n^2 h_n)^{-1/2} \sum_{j=1}^{n} h_j^{1/2} \int_{-\infty}^{\infty} f(x)J(x,h_j)dx$$

$$- 2(n^2 h_n)^{-1/2} r_n \int_{-\infty}^{\infty} f^2(x)dx + \int_{-\infty}^{\infty} f^2(x)dx + 0((nh_n)^{-1}),$$

where $J(x,h) = \int_{-\infty}^{\infty} K(t)[f(x-th)-f(x)]\ dt$. Then

$$MISE\ [f_{n_2}^*(x)] = (nh_n)^{-1} \int_{-\infty}^{\infty} K^2(t)dt$$

$$+ 2(n^2 h_n)^{-1} \sum_{1=i<j}^{n} (h_i h_j)^{-1/2} \int_{-\infty}^{\infty} J(x,h_i)J(x,h_j)dx$$

$$- 2(n^2 h_n)^{-1/2}[1 - r_n(n^2 h_n)^{-1/2}] \sum_{j=1}^{n} h_j^{1/2} \int_{-\infty}^{\infty} f(x)J(x,h_j)dx$$

$$+ [1 - (n^2 h_n)^{-1/2} r_n]^2 \int_{-\infty}^{\infty} f^2(x)dx + 0((n\ h_n)^{-1}).$$

By using Hölder's inequality we get

$$MISE\,[f^*_{n_2}(x)] \leq (n\,h_n)^{-1} \,||\, K(t)\,||^2 + [1-(n^2h_n)^{-1/2}r_n]^2 \,||\, f(x)\,||^2$$
$$+\,2(n^2h_n)^{-1} \sum_{1=i<j}^{n} (h_ih_j)^{+1/2} \,||\, J(x,h_i)\,||\,||\, J(x,h_j)\,|| + 0((n\,h_n)^{-1}). \qquad (7)$$

By using Taylor series-like expansions for f satisfying conditions of our theorem

a) For $b_{ij_1-1} < x < b_{ij_1}$, $b_{ij_2-1} < x-th < b_{ij_2}$, $1 \leq j_1 \leq l_i+1$, $1 \leq j_2 \leq l_i+1$, $i = 1,2,\cdots,k+1$

$$f(x-th)-f(x)+th\,f'(x)-h^2\int_0^t (t-s)f''(x-sh)ds$$
$$= \begin{cases} 0 & , \quad j_1 = j_2 \\ \sum\limits_{r=j_1}^{j_2-1} (b_{ir}-x+th)\Delta_{ir} & , \quad j_1 < j_2 \\ \sum\limits_{r=j_2}^{j_1-1} (b_{ir}-x+th)\Delta_{ir} & , \quad j_1 > j_2 \end{cases}$$

b) For $b_{i_1j_1-1} < x < b_{i_1j_1}$, $b_{i_2j_2-1} < x-th < b_{i_2j_2}$, $1 \leq j_1 \leq 1_{i_1}+1$, $1 \leq j_2 \leq 1_{i_2}+1$, $1 \leq i_1 \leq k+1$, $1 \leq i_2 \leq k+1$, $i_1 \neq i_2$,

$$f(x-th)-f(x)+th\,f'(x)-h^2\int_0^t (t-s)f''(x-sh)ds$$

$$= \begin{cases} \text{(i)} \quad \sum\limits_{r=j_1}^{l_{i_1}+1} (b_{i_1r}-x+th)\Delta_{i_1r} + \sum\limits_{r=i_1+1}^{i_2-1}\sum\limits_{r'=1}^{l_\ell+1} (b_{rr'}-x+th)\Delta_{rr'}, \\ \text{(ii)} \quad +\sum\limits_{r=j_2}^{j_2-1} (b_{i_2r}-x-th)\Delta_{i_2r} - \sum\limits_{\mu=i_2}^{i_2-1} \delta_\mu \text{ if } i_1 < i_2 \\ \text{(iii)} \quad -\sum\limits_{r=j_2}^{l_{i_2}+1} (b_{i_2r}-x+th)\Delta_{i_2r} - \sum\limits_{r=i_2+1}^{i_1-1}\sum\limits_{r'=1}^{l_\ell+1} (b_{rr'}-x+th)\Delta_{rr} \\ \quad\quad -\sum\limits_{r=1}^{j_1-1} (b_{i_1r}-x+th)\Delta_{i_1r} + \sum\limits_{\mu=i_2}^{i_1-1} \delta_\mu \text{ if } i_1 > i_2 \end{cases}$$

Let

$$G(x,h) \;=\; h^2\int_{-\infty}^{\infty} K(t)\int_0^t (t-s)f''(x-sh)dsdt, \;-\infty < x < \infty,$$

$$
\begin{aligned}
H_{ij1}(x,h) &= \sum_{r=j}^{l_i+1} \Delta_{ir} \int_{-\infty}^{\frac{x-b_{ir}}{h}} (b_{ir} - x + th)K(t)dt \\
&\quad + \sum_{r=i+1}^{k+1} \sum_{r'=1}^{l_{\ell+1}} \Delta_{rr'} \int_{-\infty}^{\frac{x-b_{rr'}}{h}} (b_{rr'} - x + th)K(t)dt, \\
H_{ij2}(x,h) &= -\sum_{r=1}^{j-1} \Delta_{ir} \int_{\frac{x-b_{ir}}{h}}^{\infty} (b_{ir} - x + th)K(t)dt \\
&\quad - \sum_{r=1}^{i-1} \sum_{r'=1}^{l_{\ell+1}} \Delta_{rr'} \int_{\frac{x-b_{rr'}}{h}}^{\infty} (b_{rr'} - x + th)K(t), \\
&\quad b_{ij-1} < x < b_{ij};\ j = 1,2,\cdots,l_{i+1}\ ,\ i = 1,2,\cdots,k+1, \\
V_{i1}(x,h) &= \sum_{\mu=1}^{i-1} \delta_\mu \{1 - W(\frac{x-a_\mu}{h})\},\ a_{i-1} < x < a_i,\ i = 1,\cdots,k+1, \\
V_{i2}(x,h) &= -\sum_{\mu=i}^{k} \delta_\mu W(\frac{x-a_\mu}{h}),\ a_{i-1} < x < a_i,\ i = 1,\cdots,k.
\end{aligned}
$$

Where $W(y) = \int_{-\infty}^{y} K(t)dt,\ -\infty < y < \infty$
Further let, for $j = 1,\cdots,l_i+1,\ i = 1,\cdots,k+1$

$$
\begin{aligned}
H_{ij}(x,h) &= H_{ij1}(x,h) + H_{ij2}(x,h) \text{ and} \\
V_i(x,h) &= V_{i1}(x,h) + V_{i2}(x,h).
\end{aligned}
$$

Lemma 1.

$$\lim_{h\to 0} \frac{1}{h^4} \int_{-\infty}^{\infty} G(x,h)dx = \frac{1}{4}(\int_{-\infty}^{\infty} t^2K(t)dt)^2 \cdot \int_{-\infty}^{\infty} (f''(x)^2dx.$$

Lemma 2.

$$\lim_{h\to 0} \frac{1}{h^3} \int_{b_{ij-1}}^{b_{ij}} H_{ij}^2(x,h)dx = (\Delta_{ij}^2 + \Delta_{ij-1}^2) \int_{-\infty}^{0} \{\int_{-\infty}^{y} (y-t)K(t)dt\}^2 dy,$$

$$1 \le j \le 1_{i+1},\ i = 1,\cdots,k+1.$$

Lemma 3.

$$\lim_{h\to 0} \frac{1}{h} \int_{a_{i-1}}^{a_i} V_i^2(x,h)dx = (\delta_i^2 + \delta_{i-1}^2) \int_0^{\infty} (1 - W(y))^2 dy,\ i = 1,\cdots,k+1.$$

Lemma 4.

$$
\begin{aligned}
&\lim_{h\to 0} \frac{1}{h^3} \int_{b_{ij-1}}^{b_{ij}} G(x,h)H_{ij}(x,h)dx = \lim_{h\to 0} \frac{1}{h} \int_{a_{i-1}}^{a_i} G(x,h)V_i(x,h)dx \\
&= \lim_{h\to 0} \frac{1}{h} \int_{b_{ij-1}}^{b_{ij}} H_{ij}(x,h)V_i(x,h)dx = 0, 1 \le j \le 1_{i=1},\ i = 1,\cdots,k+1
\end{aligned}
$$

Proofs of these lemmas given by Eden, C.V. [4].

We can see that

$$\begin{aligned}
&\int_{-\infty}^{\infty} J^2(x,h)dx \\
&= \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty} K(t)[f(x-th)-f(x)]dt\}^2dx \\
&= \sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}}\{G(x,h)+H_{ij}(x,h)+V_i(x,h)\}^2dx \\
&= \sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} G^2(x,h)dx + \sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} H_{ij}^2(x,h)dx \\
&\quad + \sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} V_i^2(x,h)dx + 2\sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} G(x,h)H_{ij}(x,h)dx \\
&\quad + 2\sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} G(x,h)V_i(x,h)dx + 2\sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} H_{ij}(x,h)V_i(x,h)dx \\
&= \int_{-\infty}^{\infty} G^2(x,h)dx + \sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} H_{ij}^2(x,h)dx \\
&\quad + \sum_{i=1}^{k+1}\int_{a_{i-1}}^{a_i} V_i^2(x,h)dx + 2\sum_{i=1}^{k+1}\sum_{i=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} G(x,h)H_{ij}(x,h)dx \\
&\quad + 2\sum_{i=1}^{k+1}\int_{a_{i-1}}^{a_i} G(x,h)V_i(x,h)dx + 2\sum_{i=1}^{k+1}\sum_{j=1}^{\ell_{i+1}}\int_{b_{ij-1}}^{b_{ij}} H_{ij}(x,h)V_i(x,h)dx.
\end{aligned}$$

Then from lemmas (1),(2),(3) and (4), we get

$$\begin{aligned}
\int_{-\infty}^{\infty} J^2(x,h)dx &= \frac{h^4}{4}(\int_{-\infty}^{\infty} t^2K(t)dt)^2\int_{-\infty}^{\infty}(f''(x))^2dx + 0(h^4) \\
&\quad + h^3\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}(\Delta_{ij}^2+\Delta_{ij-1}^2)\int_{-\infty}^{0}\{\int_{-\infty}^{y}(y-t)K(t)dt\}^2dy \\
&\quad + 0(h^3) + h\sum_{i=1}^{k+1}(\delta_i^2+\delta_{i-1}^2)\int_0^{\infty}(1-W(y))^2dy + 0(h).
\end{aligned}$$

But from the fact that

$$\sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\Delta_{ij-1}^2 = \sum_{i=1}^{k+1}\sum_{j=1}^{l_i+1}\Delta_{ij}^2 = \Delta,$$

and

$$\sum_{i=1}^{k+1} \delta_{i-1}^2 = \sum_{i=1}^{k+1} \delta_i^2 = \delta,$$

We have

$$\int_{-\infty}^{\infty} J^2(x, h_i)dx = \frac{1}{4}h_j^4 \parallel tK^{1/2}(t) \parallel^4 \cdot \parallel f''(x) \parallel^2$$
$$+ 0(h_j^4) + 2h_j^3\Delta \int_{-\infty}^{0} (\int_{-\infty}^{y} (y-t)K(t)dt)^2 dy + 0(h_j^3)$$
$$+ 2h_j\delta \int_0^{\infty} (1 - W(y))^2 dy + 0(h_j). \quad (8)$$

From (7) and (8) we get

$$MISE\ [f_{n_2}^*(x)] \leq (nh_n)^{-1} \parallel K(t) \parallel^2 + 0(nh_n)^{-1} + [1 - (n^2h_n)^{-1/2}r_n]^2 \parallel f(x) \parallel^2$$
$$+ 2(nh_n)^{-1}[\frac{1}{4} \parallel tK^{1/2}(t) \parallel^4 \cdot \parallel f'' \parallel^2 (\sum_{j=1}^{n} h_j^5) + 0(h_j^5)$$
$$+ 2\Delta(\sum_{j=1}^{n} h_j^4) \int_{-\infty}^{0} \{\int_{-\infty}^{y} (y-t)K(t)dt\}^2 dy$$
$$+ 0(h_j^4) + 2\delta \int_0^{\infty} \{1 - W(y)\}^2 dy (\sum_{j=1}^{n} h_j^2) + 0(h_j^2).$$

Which completes the proof of the theorem.

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