TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 3, Autumn 1991

LOCALLY NOETHERIAN LATTICE MODULES

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Let L be a multiplicative lattice and let M be a L-module. M is said to be Noetherian if M satisfies the ascending chain condition, is modular, and is principally generated (PG-lattice). If L is a Noetherian L-module, then L will be called a Noetherian lattice.

Recall that M is a K-lattice ([6], Definition 12) if it is a CG-lattice and for any compact element h of L and any compact element H of M, the element h. H is compact. And recall M is a R-lattice ([6], Definition 13) if it is a PG-lattice and every principal element of M is compact.

Let L and M be K-lattices. A L-module M is said to be locally Noetherian if M_p is a Noetherian L_p-module for each maximal element p of L. It is well known that if M is locally Noetherian, then M need not be Noetherian.

The purpose of this paper is to characterize those locally Noetherian lattice modules which are also Noetherian. So Theorem (2-3) shows that a locally Noetherian lattice module, in which the annihilator of any element of M is contained in only finitely many maximal elements of L, is Noetherian.

The afore-mentioned condition on the annihilatores of elements of locally Noetherian modules can be simplified for locally Noetherian lattices, by restricting it to the annihilatores of their prime elements (Theorem (2-7)). Then our lattice results can be applied to commutative rings and modules; and thus we obtain new conditions which are necessary for locally Noetherian rings and modules to be Noetherian.

Throught this paper L will denote a multiplicative lattice and M will denote a lattice module over L. Our notation and terminology are essentially that of ([3], [5], [6]) and [7].

1. General Properties.

Lemma (1-1). *If the greatest element I of L is compact, then for every element $b \neq I$ in L there is a maximal element p of L such that $b \leq p$.*

Proof. This is essentially the same as given for Lemma (1-1) in [7].

Proposition (1-2). *Let L be a K-lattice in which the greatest element I is compact.*

Received August 8, 1990.

Let M be a K-lattice and let $B, D \in M$. If $[B] = [D]$ in M_p, for every maximal element p of L, then $B = D$.

Proof. This follows from Proposition (5-3) in [6] and Lemma (1-1).

Proposition (1-3). *Let L be a K-lattice in which the greatest element I is compact. Let M be a K-lattice and let B be an element of M. If $(O : B) \not\leq p$, then $[B] = [O]$ in M_p for every prime element p of L.*

Proof. Let p be a prime element of L and let $[B] \neq [O]$ in L_p. Then there exists a compact element H of M such that $H \leq B$ and $[H] \neq [O]$ in M_p. This implies that $([O] : [H]) \neq [I]$ in L_p. By Lemma (9-2) in [6] and Lemma (1-1) we get that $([O] : [H]) = [O : H] \leq p$. This implies that $(O : H) \leq S(p) = p$. So $(O : B) \leq p$ since $(O : B) \leq (O : H)$, a contradication. Thus $[B] = [O]$.

Lemma (1-4). *Let M be a K-lattice. Then the following conditions are equivalent:*
(i) *M satisfies the ascending chain condition.*
(ii) *Every element of M is compact.*

Proof. The proof is obvious.

Lemma (1-5). *Let M be a R-lattice and let B be an element of M. Then B is compact, if and only if, B is finitely generated.*

Proof. It follows from the definition of the R-lattice.

Proposition (1-6). *Let M be a R-lattice. Then the following conditions are equivalent:*
(i) *M satisfies the ascending chain condition.*
(ii) *Every element of M is finitely generated.*
(iii) *Every element of M is compact.*

Proof. Obviously (i) implies (ii) since every element of M is a join of principal elements. To see that (ii) implies (i), let $B_1 \leq B_2 \leq \ldots$ be an ascending chain of elements of M. Then the element $B = \bigvee_{i=1}^{\infty} B_i$ of M is finitely generated. Let $B = A_1 \vee \ldots \vee A_n$ where A_j $(1 \leq j \leq n)$ is a principal element of M. Since a finite join of principal elements is compact we get that B is compact and hence $B = B_k$ for some integer k. Therefore, M satisfies the ascending chain condition. Lemma (1-4) shows that (i) and (ii) are equivalent.

2. Locally Noetherian lattice modules.

It is well known that if R is a locally Noetherian commutative ring, then R need not be Noetherian (Example (2-2) in [4]). So the multiplicative lattice $L(R)$ of ideals of R is a locally Noetherian lattice, but need not be Noetherian. It follows that a locally Noetherian lattice module M over L need not be Noetherian. The following Theorems and Proposition give some criterions which are necessary for locally Noetherian lattice modules to be Noetherian.

Theorem (2-1). *Let L be a K-lattice in which the greatest element I is compact. Let M be a K-lattice and let B be an element of M, whose annihilator is contained in only finitely many maximal elements $p_1, \ldots, p_n$ of L. If $[B]$ is a compact element in M_{p_i} over L_{p_i} for $1 \leq i \leq n$, then B is a compact element in M.*

Proof. For $i = 1, \cdots, n$, let p_i be a maximal element of L containing $(O : B)$. By assumption $[B]$ is a compact element in M_{p_i}. Therefore there exists a compact element H_i of M such that $H_i \leq B$ and $[H_i] = [B]$ in M_{p_i}. Suppose that $H = H_1 \vee \ldots \vee H_n$, then H is compact and $[H] = [B]$ in M_{p_i} for every maximal element p_i $(1 \leq i \leq n)$. Furthermore if p is a maximal element of L such that $(O : B) \not\leq p$, then by Proposition (1-3) we get that $[O] = [B]$ in M_p over L_p. Therefore $[H] = [B]$ in M_p for all maximal elements p of L. Thus by Proposition (1-2) we have that $H = B$ and hence B is compact.

Proposition (2-2). *Let L be a K-lattice in which the greatest element I is compact. Let M be a R-lattice and let B be an element of M, whose annihilator is contained in only finitely many maximal elements $p_1, \ldots, p_n$ of L. If B is finitely generated in M_{p_i} for $1 \leq i \leq n$, then B is finitely generated.*

Proof. Since M_p is a R-lattice, we get that $[B]$ is a finitely generated element in M_p, if and only if, it is compact. Therefore Theorem (2-1) shows that B is compact in M, and by Proposition (1-6) it follows that B is finitely generated.

Theorem (2-3). *Let L be a K-lattice in which the greatest element I is compact and let M be a R-lattice. If M is locally Noetherian and the annihilator of any element of M is contained in only finitely many maximal elements $p_1, \ldots, p_n$ of L, then M is Noetherian.*

Proof. Let B be an element of M. By assumption the element $[B]$ is finitely generated in M_{p_i} over L_{p_i} $(1 \leq i \leq n)$. Hence by Proposition (2-2) it follows that M satisfies the ascending chain condition. Furthermore Corollary (7-3) in [6] shows that M is modular.

Corollary (2-4). *Let L be a semi-local K-lattice in which the greatest element I is compact and let M be a R-lattice. If M is a locally Noetherian module, then M is Noetherian.*

Proof. This follows from Theorem (2-3) and from the definition of a semi-local lattice.

Theorem (2-5). *Let L be a locally Noetherian R-lattice. If the annihilator of any element of L is contained in only finitely many maximal elements of L, then L is Noetherian.*

Proof. This follows from Theorem (2-3) by regarding L as an L-module.

Now we shall simplify the condition related to annihilators of elements of L.

Theorem (2-6). *Let L be a modular R-lattice. If any prime element of L is finitely generated, then L satisfies the ascending chain condition.*

Proof. By Proposition (1-6), it is sufficient to prove that any element of L is finitely generated. Let T be the set of all elements of L which are not finitely generated. If $T \neq \phi$, then by Zorn's Lemma, T contains a maximal element p. Since p can not be a prime element in L, then there exist two principal elements a_1, a_2 of L such that $a_1 \cdot a_2 \leq p$, $a_1 \not\leq p$ & $a_2 \not\leq p$. This implies that $(P \vee a_1) > P$ and $(P : a_1) > P$. Therefore $(P \vee a_1)$ and $(P : a_1)$ are finitely generated. It follows that $(P : a_1) \cdot a_1$ is finitely generated. Since M is a join of principal elements and $P \vee a_1$ is finitely generated there exists a finitely generated element b of L such that $b \leq P$ and $b \vee a_1 = P \vee a_1$. By the modularity we find that:

$$P = P \wedge (P \vee a_1) = P \wedge (b \vee a_1) = b \vee (P \wedge a_1) = b \vee (P : a_1) \cdot a_1$$

This implies that $P \notin T$, a contradication.

Theorem (2-7). *Let L be a locally Noetherian R-lattice. If the annihilator of any prime element of L is contained in only finitely many maximal elements of L, then L is Noetherian.*

Proof. Let q be a prime element of L, and let p be a maximal element which is containing the annihilator of q. By assumption L_p is a Noetherian lattice and hence $[q]$ is finitely generated in L_p. Proposition (2-2) shows that q is finitely generated in L and by Theorem (2-6) L satisfies the ascending chain condition. Furthermore L is modular by Corollary (7-3) in [6], which completes the proof.

When viewed in the context of ring theory these results translate to the following:

Proposition (2-8). *Let R be a commutative ring and let M be a locally Noetherian R-module. If the annihilator of any submodule of M is contained in only finitely many maximal ideals of R, then M is Noetherian.*

Proof. Let $L(R)$ be the lattice of ideals of R and let $L(M)$ be the lattice of submodules of M. Then $L(R)$ is a multiplicative lattice and $L(M)$ is a lattice module over $L(R)$ with known structure operations on ideals and submodules. Each of $L(R)$ and $L(M)$ is satisfying the conditions of the Theorem (2-3) and hence the module $L(M)$ is Noetherian. It means that M is Noetherian.

Corollary (2-9). *Let R be a semi-local ring and let M be a R-module. If M is a locally Noetherian module, then M is Noetherian.*

Proof. It is obvious.

Proposition (2-10). *Let R be a locally Noetherian ring. If the annihilator of any prime ideal of R is contained in only finitely many maximal ideals of R, then R is Noetherian.*

Proof. This follows by applying Theorem (2-7) on the multiplicative lattice $L(R)$.

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