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NEW INEQUALITIES OF OSTROWSKI AND GRÜSS TYPE FOR TRIPLE INTEGRALS

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Abstract. In the present paper we establish new inequalities of Ostrowski and Grüss type for triple integrals involving three functions and their partial derivatives. The discrete Ostrowski and Grüss type inequalities for triple sums are also given.

1. Introduction

In a well known paper of 1938, A.Ostrowski [6] proved the following inequality (see also [5, p.469]):

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\, \|f'\|_\infty, \tag{1.1}$$

for all $x \in [a,b]$, where $f : [a,b] \subseteq R \to R$ is continuous on $[a,b]$ and differentiable on (a,b), whose derivative $f' : (a,b) \to R$ is bounded on (a,b), i.e. $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$.

In a celebrated paper of 1935, G Grüss [3] proved the following inequality (see also [4, p.296]):

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \right| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{1.2}$$

provided that f and g are two integrable functions on $[a,b]$ such that

$$\phi \le f(x) \le \Phi, \gamma \le g(x) \le \Gamma,$$

for all $x \in [a,b]$, where $\Phi, \phi, \Gamma, \gamma$ are real constants.

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In the past few years, an enormous amount of attention has been given to the above inequalities and numerious generalizations, extensions and variants have appeared in the literature, see [1, 2, 4, 5, 7, 8, 9, 10, 11, 12] and the references cited therein. The main purpose of this paper is to establish new inequalities for triple integrals, similar to those in (1.1) and (1.2) involving three functions and their partial derivatives. The discrete inequalities of Ostrowski and Grüss type for triple sums are also given. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

2. Statement of results

In what follows R denotes the set of real numbers, let $[a_i, b_i]$ $(a_i < b_i)$ for $i = 1, 2, 3$ are the given subsets of R and $H = \prod_{i=1}^{3} [a_i, b_i]$. The partial derivatives of a function $e = e(x, y, z) : H \to R$ are denoted by $D_1 e = \frac{\partial}{\partial x} e$, $D_2 e = \frac{\partial}{\partial y} e$, $D_3 e = \frac{\partial}{\partial z} e$, $D_1 D_2 e = \frac{\partial^2}{\partial x \partial y} e$, $D_2 D_3 e = \frac{\partial^2}{\partial y \partial z} e$, $D_3 D_1 e = \frac{\partial^2}{\partial z \partial x} e$ and $D_3 D_2 D_1 e = \frac{\partial^3}{\partial z \partial y \partial x} e$. We denote by $F(H)$ the class of continuous functions $e : H \to R$ for which $D_1 e$, $D_2 e$, $D_3 e$, $D_1 D_2 e$, $D_2 D_3 e$, $D_3 D_1 e$, $D_3 D_2 D_1 e$ exist and are continuous on H. Let N denote the set of natural numbers, $A = \{1, 2, \ldots, a+1\}$, $B = \{1, 2, \ldots, b+1\}$, $C = \{1, 2, \ldots, c+1\}$, for $a, b, c \in N$ and $E = A \times B \times C$. For a function $e = e(k, m, n) : E \to R$ we define the difference operators by $\Delta_1 e = e(k+1, m, n) - e(k, m, n)$, $\Delta_2 e = e(k, m+1, n) - e(k, m, n)$, $\Delta_3 e = e(k, m, n+1) - e(k, m, n)$, $\Delta_1 \Delta_2 e = \Delta_1 (\Delta_2 e)$, $\Delta_2 \Delta_3 e = \Delta_2 (\Delta_3 e)$, $\Delta_3 \Delta_1 e = \Delta_3 (\Delta_1 e)$, $\Delta_3 \Delta_2 \Delta_1 e = \Delta_3 (\Delta_2 \Delta_1 e)$. We denote by $G(E)$ the class of functions $e = e(k, m, n) : E \to R$ for which $\Delta_1 e$, $\Delta_2 e$, $\Delta_3 e$, $\Delta_1 \Delta_2 e$, $\Delta_2 \Delta_3 e$, $\Delta_3 \Delta_1 e$, $\Delta_3 \Delta_2 \Delta_1 e$ exist on E. We assume that $e(k, m, n) = 0$ for $(k, m, n) \notin E$ and also use the usual convention that, the empty sum is taken to be zero.

First we give the following notations used to simplify the details of presentation.

For $i = 1, 2, 3$; $a_i, b_i \in R$, $(x, y, z), (r, s, t) \in H$ and some suitable functions $p, f, g, h : H \to R$, we set

$$\Delta = (b_1 - a_1) \times (b_2 - a_2) \times (b_3 - a_3),$$

$$I[p] = \int_r^x \int_s^y \int_t^z D_3 D_2 D_1 p(u, v, w)\, dw dv du,$$

$$\begin{aligned} J[p] = {} & (b_2 - a_2)(b_3 - a_3) \int_{a_1}^{b_1} p(r, y, z)\, dr \\ & + (b_1 - a_1)(b_3 - a_3) \int_{a_2}^{b_2} p(x, s, z)\, ds \\ & + (b_1 - a_1)(b_2 - a_2) \int_{a_3}^{b_3} p(x, y, t)\, dt, \end{aligned}$$

$$L[p] = (b_3 - a_3)\int_{a_1}^{b_1}\int_{a_2}^{b_2} p(r,s,z)\,dsdr + (b_2 - a_2)\int_{a_1}^{b_1}\int_{a_3}^{b3} p(r,y,t)\,dtdr + (b_1 - a_1)\int_{a_2}^{b_2}\int_{a_3}^{b_3} p(x,s,t)\,dtds,$$

$$A(f,g,h;J,L;\Delta)(x,y,z) = f(x,y,z)\,g(x,y,z)\,h(x,y,z) - \frac{1}{3\Delta}\left[g(x,y,z)\,h(x,y,z)\left\{J[f] - L[f] + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(r,s,t)\,dtdsdr\right\}\right.$$
$$+h(x,y,z)\,f(x,y,z)\left\{J[g] - L[g] + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(r,s,t)\,dtdsdr\right\}$$
$$\left.+f(x,y,z)\,g(x,y,z)\left\{J[h] - L[h] + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} h(r,s,t)\,dtdsdr\right\}\right],$$

$$B(f,g,h;I)(x,y,z) = g(x,y,z)\,h(x,y,z)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} I[f]\,dtdsdr + h(x,y,z)\,f(x,y,z)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} I[g]\,dtdsdr + f(x,y,z)\,g(x,y,z)\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} I[h]\,dtdsdr,$$

$$T(f,g,h;J,L;\Delta) = \frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)g(x,y,z)\,h(x,y,z)\,dzdydx$$
$$-\frac{1}{3\Delta^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\{g(x,y,z)\,h(x,y,z)\,(J[f] - L[f])$$
$$+h(x,y,z)\,f(x,y,z)\,(J[g] - L[g])$$
$$+f(x,y,z)\,g(x,y,z)\,(J[h] - L[h])\}\,dzdydx$$
$$-\frac{1}{3}\left[\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(x,y,z)\,h(x,y,z)dzdydx\right)\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)dzdydx\right)\right.$$
$$+\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} h(x,y,z)\,f(x,y,z)dzdydx\right)\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} g(x,y,z)dzdydx\right)$$
$$\left.+\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x,y,z)\,g(x,y,z)dzdydx\right)\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} h(x,y,z)dzdydx\right)\right].$$

For $a, b, c \in N$ and some suitable functions $p, f, g, h : E \to R$, we set

$$l = abc$$

$$\bar{I}\,[p] = \sum_{u=r}^{k-1} \sum_{v=s}^{m-1} \sum_{w=t}^{n-1} \Delta_3 \Delta_2 \Delta_1 p\,(u, v, w)\,,$$

$$\bar{J}\,[p] = bc \sum_{r=1}^{a} p\,(r, m, n) + ca \sum_{s=1}^{b} p\,(k, s, n) + ab \sum_{t=1}^{c} p\,(k, m, t),$$

$$\bar{L}\,[p] = c \sum_{r=1}^{a} \sum_{s=1}^{b} p\,(r, s, n) + b \sum_{r=1}^{a} \sum_{t=1}^{c} p\,(r, m, t) + a \sum_{s=1}^{b} \sum_{t=1}^{c} p\,(k, s, t),$$

$$P\left(f, g, h; \bar{J}, \bar{L}; l\right)(k, m, n) = f\,(k, m, n)\, g\,(k, m, n)\, h\,(k, m, n)$$
$$-\frac{1}{3l}\left[g\,(k, m, n)\, h\,(k, m, n) \left\{ \bar{J}\,[f] - \bar{L}\,[f] + \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} f\,(r, s, t) \right\} \right.$$
$$+ h\,(k, m, n)\, f\,(k, m, n) \left\{ \bar{J}\,[g] - \bar{L}\,[g] + \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} g\,(r, s, t) \right\}$$
$$\left. + f\,(k, m, n)\, g\,(k, m, n) \left\{ \bar{J}\,[h] - \bar{L}\,[h] + \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} h\,(r, s, t) \right\} \right],$$

$$Q\left(f, g, h; \bar{I}\right)(k, m, n)$$
$$= g\,(k, m, n)\, h\,(k, m, n) \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} \bar{I}\,[f]$$
$$+ h\,(k, m, n)\, f\,(k, m, n) \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} \bar{I}\,[g]$$
$$+ f\,(k, m, n)\, g\,(k, m, n) \sum_{r=1}^{a} \sum_{s=1}^{b} \sum_{t=1}^{c} \bar{I}\,[h],$$

$$M\left(f, g, h; \bar{J}, \bar{L}; l\right) = \frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} f\,(k, m, n)\, g\,(k, m, n)\, h\,(k, m, n)$$
$$-\frac{1}{3l^2} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} \left\{ g\,(k, m, n)\, h\,(k, m, n) \left(\bar{J}\,[f] - \bar{L}\,[f] \right) \right.$$
$$\left. + h\,(k, m, n)\, f\,(k, m, n) \left(\bar{J}\,[g] - \bar{L}\,[g] \right) + f\,(k, m, n)\, g\,(k, m, n) \left(\bar{J}\,[h] - \bar{L}\,[h] \right) \right\}$$
$$-\frac{1}{3}\left[\left(\frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} g\,(k, m, n) h\,(k, m, n) \right) \left(\frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} f\,(k, m, n) \right) \right.$$

$$+\left(\frac{1}{l}\sum_{k=1}^{a}\sum_{m=1}^{b}\sum_{n=1}^{c}h\left(k,m,n\right)f\left(k,m,n\right)\right)\left(\frac{1}{l}\sum_{k=1}^{a}\sum_{m=1}^{b}\sum_{n=1}^{c}g\left(k,m,n\right)\right)$$
$$+\left(\frac{1}{l}\sum_{k=1}^{a}\sum_{m=1}^{b}\sum_{n=1}^{c}f\left(k,m,n\right)g\left(k,m,n\right)\right)\left(\frac{1}{l}\sum_{k=1}^{a}\sum_{m=1}^{b}\sum_{n=1}^{c}h\left(k,m,n\right)\right)\Bigg].$$

Our main results are given in the following theorems.

Theorem 1. *Let $f,g,h\in F\left(H\right)$. Then*

$$\left|A\left(f,g,h;J,L;\Delta\right)\left(x,y,z\right)\right|\leq\frac{1}{3\Delta}B\left(\left|f\right|,\left|g\right|,\left|h\right|;\left|I\right|\right)\left(x,y,z\right),\tag{2.1}$$

for all $(x,y,z)\in H$.

Remark 1. If we take $h(x,y,z)=1$ and hence $I[h]=0$ in Theorem 1, then by elementary calculations we get

$$\Bigg|f\left(x,y,z\right)g\left(x,y,z\right)-\frac{1}{2\Delta}\left[g\left(x,y,z\right)\left\{J\left[f\right]-L\left[f\right]+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f\left(r,s,t\right)dtdsdr\right\}\right.$$
$$\left.\left.+f\left(x,y,z\right)\left\{J\left[g\right]-L\left[g\right]+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}g\left(r,s,t\right)dtdsdr\right\}\right]\right|$$
$$\leq\frac{1}{2\Delta}\left[\left|g\left(x,y,z\right)\right|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|I\left[f\right]\right|dtdsdr+\left|f\left(x,y,z\right)\right|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|I\left[g\right]\right|dtdsdr\right],\tag{2.2}$$

for all $(x,y,z)\in H$. Further, by taking $g(x,y,z)=1$ and hence $I[g]=0$ in (2.2) and by simple calculations we get

$$\left|f\left(x,y,z\right)-\frac{1}{\Delta}\left[J\left[f\right]-L\left[f\right]+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}f\left(r,s,t\right)dtdsdr\right]\right|$$
$$\leq\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|I\left[f\right]\right|dtdsdr,\tag{2.3}$$

for all $(x,y,z)\in H$.

Theorem 2. *Let $f,g,h\in F\left(H\right)$. Then*

$$\left|T\left(f,g,h;J,L;\Delta\right)\right|\leq\frac{1}{3\Delta^2}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}B\left(\left|f\right|,\left|g\right|,\left|h\right|;\left|I\right|\right)\left(x,y,z\right)dzdydx.\tag{2.4}$$

Remark 2. In the special case, when $h(x,y,z)=1$ and hence $I[h]=0$, it is easy to observe that the inequality obtained in (2.4) reduces to the following Grüss type inequality

$$\Bigg|\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b}\int_{a_3}^{b_3}f\left(x,y,z\right)g\left(x,y,z\right)dzdydx$$

$$-\frac{1}{2\Delta^2}\int_{a_1}^{b_1}\int_{a_2}^{b}\int_{a_3}^{b_3}\left\{g\left(x,y,z\right)\left(J\left[f\right]-L\left[f\right]\right)+f\left(x,y,z\right)\left(J\left[g\right]-L\left[g\right]\right)\right\}dzdydx$$
$$-\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b}\int_{a_3}^{b_3}f\left(x,y,z\right)dzdydx\right)\left(\frac{1}{\Delta}\int_{a_1}^{b_1}\int_{a_2}^{b}\int_{a_3}^{b_3}g\left(x,y,z\right)dzdydx\right)\Bigg|$$
$$\leq\frac{1}{2\Delta^2}\int_{a_1}^{b_1}\int_{a_2}^{b}\int_{a_3}^{b_3}\Bigg[\left|g\left(x,y,z\right)\right|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|I\left[f\right]\right|dtdsdr$$
$$+\left|f\left(x,y,z\right)\right|\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3}\left|I\left[g\right]\right|dtdsdr\Bigg]dzdydx. \tag{2.5}$$

For similar results, see [8, 10].

The discrete inequalities of the forms (2.1) and (2.4) are given in the following theorems.

Theorem 3. *Let $f,g,h\in G\left(E\right)$. Then*

$$\left|P\left(f,g,h;\bar{J},\bar{L};l\right)\left(k,m,n\right)\right|\leq\frac{1}{3l}Q\left(\left|f\right|,\left|g\right|,\left|h\right|;\left|I\right|\right)\left(k,m,n\right), \tag{2.6}$$

for all $(k,m,n)\in E$.

Remark 3. Taking $h(k,m,n)=1$ and hence $\bar{I}\left[h\right]=0$ in Theorem 3 and by simple computations , it is easy to see that the inequality (2.6) reduces to

$$\Bigg|f\left(k,m,n\right)g\left(k,m,n\right)-\frac{1}{2l}\Bigg[g\left(k,m,n\right)\left\{\bar{J}\left[f\right]-\bar{L}\left[f\right]+\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}f\left(r,s,t\right)\right\}$$
$$+f\left(k,m,n\right)\left\{\bar{J}\left[g\right]-\bar{L}\left[g\right]+\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}g\left(r,s,t\right)\right\}\Bigg]\Bigg|$$
$$\leq\frac{1}{2l}\left[\left|g\left(k,m,n\right)\right|\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}\left|\bar{I}\left[f\right]\right|+\left|f\left(k,m,n\right)\right|\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}\left|\bar{I}\left[g\right]\right|\right], \tag{2.7}$$

for all $(k,m,n)\in E$. Further, by taking $g(k,m,n)=1$ and hence $\bar{I}\left[g\right]=0$ in (2.7) and by simple computation we get

$$\left|f\left(k,m,n\right)-\frac{1}{l}\left[\bar{J}\left[f\right]-\bar{L}\left[f\right]+\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}f\left(r,s,t\right)\right]\right|$$
$$\leq\frac{1}{l}\sum_{r=1}^{a}\sum_{s=1}^{b}\sum_{t=1}^{c}\left|\bar{I}\left[f\right]\right|, \tag{2.8}$$

for all $(k,m,n)\in E$. For similar results, see [7, 9].

Theorem 4. *Let $f, g, h \in G(E)$. Then*

$$\left|M\left(f, g, h; \bar{J}, \bar{L}; l\right)\right| \leq \frac{1}{3l^2} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} Q\left(|f|, |g|, |h|; \left|\bar{I}\right|\right)(k, m, n). \tag{2.9}$$

Remark 4. If we take $h(k,m,n) = 1$ and hence $\left|\bar{I}[h]\right| = 0$ in Theorem 4, then by simple computations we get the following discrete Grüss type inequality

$$\begin{aligned} &\left| \frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} f(k,m,n)\, g(k,m,n) \right. \\ &\quad - \frac{1}{2l^2} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} \left\{ g(k,m,n)\left(\bar{J}[f] - \bar{L}[f]\right) + f(k,m,n)\left(\bar{J}[g] - \bar{L}[g]\right) \right\} \\ &\quad \left. - \left(\frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} f(k,m,n) \right) \left(\frac{1}{l} \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} g(k,m,n) \right) \right| \\ &\leq \frac{1}{2l^2} \left[|g(k,m,n)| \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} \left|\bar{I}[f]\right| + |f(k,m,n)| \sum_{k=1}^{a} \sum_{m=1}^{b} \sum_{n=1}^{c} \left|\bar{I}[g]\right| \right]. \end{aligned} \tag{2.10}$$

For similar inequalities, see [9, 10, 11].

3. Proofs of Theorems 1 and 2

First, we prove the following identity

$$\begin{aligned} I[f] = f(x,y,z) &- [f(r,y,z) + f(x,s,z) + f(x,y,t)] \\ &+ [f(r,s,z) + f(r,y,t) + f(x,s,t)] - f(r,s,t), \end{aligned} \tag{3.1}$$

for $(x,y,z), (r,s,t) \in H$, where

$$I[f] = \int_r^x \int_s^y \int_t^z D_3 D_2 D_1 f(u,v,w)\, dw dv du. \tag{3.2}$$

From (3.2) it is easy to observe that

$$\begin{aligned} I[f] &= \int_r^x \int_s^y D_2 D_1 f(u,v,z)\, dv du - \int_r^x \int_s^y D_2 D_1 f(u,v,t)\, dv du \\ &= I_1[f] - I_2[f]. \end{aligned} \tag{3.3}$$

By simple computation we have

$$I_1[f] = \int_r^x \int_s^y D_2 D_1 f(u,v,z)\, dv du$$

$$= \int_r^x D_1 f(u,y,z)\,du - \int_r^x D_1 f(u,s,z)\,du$$
$$= f(x,y,z) - f(r,y,z) - f(x,s,z) + f(r,s,z)\,. \tag{3.4}$$

Similarly we have

$$I_2[f] = \int_r^x \int_s^y D_2 D_1 f(u,v,t)\,dvdu$$
$$= f(x,y,t) - f(r,y,t) - f(x,s,t) + f(r,s,z)\,. \tag{3.5}$$

Using (3.4) and (3.5) in (3.3) we get (3.1). Similarly, we have the following identities

$$I[g] = g(x,y,z) - [g(r,y,z) + g(x,s,z) + g(x,y,t)]$$
$$+ [g(r,s,z) + g(r,y,t) + g(x,s,t)] - g(r,s,t)\,, \tag{3.6}$$

$$I[h] = h(x,y,z) - [h(r,y,z) + h(x,s,z) + h(x,y,t)]$$
$$+ [h(r,s,z) + h(r,y,t) + h(x,s,t)] - h(r,s,t)\,, \tag{3.7}$$

for $(x,y,z), (r,s,t) \in H$.

Multiplying both sides of (3.1), (3.6) and (3.7) by $g(x,y,z)h(x,y,z)$, $h(x,y,z)f(x,y,z)$ and $f(x,y,z)g(x,y,z)$ respectively and adding the resulting identities we get

$$\begin{aligned}
&3f(x,y,z)\,g(x,y,z)\,h(x,y,z)\\
&\quad -g(x,y,z)\,h(x,y,z)\,\{[f(r,y,z) + f(x,s,z) + f(x,y,t)]\\
&\quad - [f(r,s,z) + f(r,y,t) + f(x,s,t)] + f(r,s,t)\}\\
&\quad -h(x,y,z)\,f(x,y,z)\,\{[g(r,y,z) + g(x,s,z) + g(x,y,t)]\\
&\quad - [g(r,s,z) + g(r,y,t) + g(x,s,t)] + g(r,s,t)\}\\
&\quad -f(x,y,z)\,g(x,y,z)\,\{[h(r,y,z) + h(x,s,z) + h(x,y,t)]\\
&\quad - [h(r,s,z) + h(r,y,t) + h(x,s,t)] + h(r,s,t)\}\\
&= g(x,y,z)\,h(x,y,z)\,I[f] + h(x,y,z)\,f(x,y,z)\,I[g]\\
&\quad +f(x,y,z)\,g(x,y,z)\,I[h]\,.
\end{aligned} \tag{3.8}$$

Integrating both sides of (3.8) with respect to (r,s,t) over H and rewriting we have

$$A(f,g,h;J,L;\Delta)(x,y,z) = \frac{1}{3\Delta} B(f,g,h;I)(x,y,z)\,. \tag{3.9}$$

From (3.9) and using the properties of modulus we get the desired inequality in (2.1). The proof of Theorem 1 is complete.

Integrating both sides of (3.9) with respect to (x,y,z) over H and rewriting we have

$$T(f,g,h;J,L;\Delta) = \frac{1}{3\Delta^2} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \int_{a_3}^{b_3} B(f,g,h;I)(x,y,z)\,dzdydx. \tag{3.10}$$

From (3.10) and using the properties of modulus we get the required inequality in (2.4). The proof of Theorem 2 is complete.

4. Proofs of Theorems 3 and 4

We first prove the following identity

$$\begin{aligned}\bar{I}[f] &= f(k,m,n) - [f(r,m,n) + f(k,s,n) + f(k,m,t)] \\ &\quad + [f(r,s,n) + f(r,m,t) + f(k,s,t)] - f(r,s,t),\end{aligned} \tag{4.1}$$

for $(k,m,n), (r,s,t) \in E$, where

$$\bar{I}[f] = \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\sum_{w=t}^{n-1} \Delta_3\Delta_2\Delta_1 f(u,v,w). \tag{4.2}$$

From (4.2) we observe that

$$\begin{aligned}\bar{I}[f] &= \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\left[\sum_{w=t}^{n-1}\{\Delta_2\Delta_1 f(u,v,w+1) - f(u,v,w)\}\right] \\ &= \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\Delta_2\Delta_1 f(u,v,n) - \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\Delta_2\Delta_1 f(u,v,t) \\ &= \bar{I}_1[f] - \bar{I}_2[f].\end{aligned} \tag{4.3}$$

By simple computation, we have

$$\begin{aligned}\bar{I}_1[f] &= \sum_{u=r}^{k-1}\sum_{v=s}^{m-1}\Delta_2\Delta_1 f(u,v,n) \\ &= \sum_{u=r}^{k-1}\left[\sum_{v=s}^{m-1}\{\Delta_1 f(u,v+1,n) - \Delta_1 f(u,v,n)\}\right] \\ &= \sum_{u=r}^{k-1}\Delta_1 f(u,m,n) - \sum_{u=r}^{k-1}\Delta_1 f(u,s,n) \\ &= \sum_{u=r}^{k-1}\{f(u+1,m,n) - f(u,m,n)\} - \sum_{u=r}^{k-1}\{f(u+1,s,n) - f(u,s,n)\} \\ &= f(k,m,n) - f(r,m,n) - f(k,s,n) + f(r,s,n).\end{aligned} \tag{4.4}$$

Similarly, we have

$$\bar{I}_2[f] = f(k,m,t) - f(r,m,t) - f(k,s,t) + f(r,s,t). \tag{4.5}$$

Using (4.4) and (4.5) in (4.3) we get (4.1). Similarly we have the following identities

$$\bar{I}[g] = g(k,m,n) - [g(r,m,n) + g(k,s,n) + g(k,m,t)]$$

$$+\left[g\left(r,s,n\right)+g\left(r,m,t\right)+g\left(k,s,t\right)\right]-g\left(r,s,t\right), \tag{4.6}$$

$$\begin{aligned}\bar{I}\left[h\right]=&\ h\left(k,m,n\right)-\left[h\left(r,m,n\right)+h\left(k,s,n\right)+h\left(k,m,t\right)\right]\\&+\left[h\left(r,s,n\right)+h\left(r,m,t\right)+h\left(k,s,t\right)\right]-h\left(r,s,t\right),\end{aligned} \tag{4.7}$$

for $(k,m,n),(r,s,t)\in E$.

Multiplying both sides of (4.1), (4.6) and (4.7) by $g(k,m,n)h(k,m,n)$, $h(k,m,n)f(k,m,n)$ and $f(k,m,n)g(k,m,n)$ respectively and adding the resulting identities we have

$$\begin{aligned}&3f\left(k,m,n\right)g\left(k,m,n\right)h\left(k,m,n\right)\\&\quad-g\left(k,m,n\right)h\left(k,m,n\right)\left\{\left[f\left(r,m,n\right)+f\left(k,s,n\right)+f\left(k,m,t\right)\right]\right.\\&\quad\left.-\left[f\left(r,s,n\right)+f\left(r,m,t\right)+f\left(k,s,t\right)\right]+f\left(r,s,t\right)\right\}\\&\quad-h\left(k,m,n\right)f\left(k,m,n\right)\left\{\left[g\left(r,m,n\right)+g\left(k,s,n\right)+g\left(k,m,t\right)\right]\right.\\&\quad\left.-\left[g\left(r,s,n\right)+g\left(r,m,t\right)+g\left(k,s,t\right)\right]+g\left(r,s,t\right)\right\}\\&\quad-f\left(k,m,n\right)g\left(k,m,n\right)\left\{\left[h\left(r,m,n\right)+h\left(k,s,n\right)+h\left(k,m,t\right)\right]\right.\\&\quad\left.-\left[h\left(r,s,n\right)+h\left(r,m,t\right)+h\left(k,s,t\right)\right]+h\left(r,s,t\right)\right\}\\&=g\left(k,m,n\right)h\left(k,m,n\right)\bar{I}\left[f\right]+h\left(k,m,n\right)f\left(k,m,n\right)\bar{I}\left[g\right]\\&\quad+f\left(k,m,n\right)g\left(k,m,n\right)\bar{I}\left[h\right]\end{aligned} \tag{4.8}$$

Summing both sides of (4.8) first with respect to t from 1 to c, then with respect to s from 1 to b and finally with respect to r from 1 to a and rewriting we have

$$P\left(f,g,h;\bar{J},\bar{L};l\right)\left(k,m,n\right)=\frac{1}{3l}Q\left(f,g,h;\bar{I}\right)\left(k,m,n\right). \tag{4.9}$$

From (4.9) and using the properties of modulus we get the required inequality in (2.6). The proof of Theorem 3 is complete.

Summing both sides of (4.9) first with respect to n from 1 to c, then with respect to m from 1 to b and finally with respect to k from 1 to a and rewriting we have

$$M\left(f,g,h;\bar{J},\bar{L};l\right)=\frac{1}{3l^2}\sum_{k=1}^{a}\sum_{m=1}^{b}\sum_{n=1}^{c}Q\left(f,g,h;\bar{I}\right)\left(k,m,n\right). \tag{4.10}$$

From (4.10) and using the properties of modulus we get the desired inequality in (2.9). The proof of Theorem 4 is complete.

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