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ON THE COMMUTATIVITY AND ANTICOMMUTATIVITY OF RINGS II*

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Abstract. It is shown that if R is any associative ring such that for each $x, y \in R$, there exist an even natural number $m(x,y)$ and an odd natural number $n(x,y)$, depending on x and y, with either $[x,y]^{m(x,y)} = [x,y]^{n(x,y)}$ or $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$, then either $[x,y]$ or $(x \circ y)$ is nilpotent for all x, y in R. Moreover, R is commutative if R has no nonzero nil right ideals.

1. Introduction

Throughout the paper, R will denote an associative ring with Jacobson radical $J(R)$. For $x, y \in R$, set $[x,y] = xy - yx$ and $(x \circ y) = xy + yx$. R is called commutative (resp. anticommutative) if $[x,y] = 0$ (resp. $(x \circ y) = 0$) for all x, y in R. R is called semiprime if for $a \in R$, $aRa = 0$ implies that $a = 0$. A right ideal A of R is called nil if for every $a \in A$, $a^n = 0$ for some positive integer $n = n(a)$. An element x in R is called periodic (resp. strongly periodic) if there exists an integer $n > 1$ (resp. an even positive integer m), with $x^n = x$ (resp. $x^m = x$). An element x of R is called $(*)$-periodic if there exist an even natural number m and an odd natural number n such that $x^m = x^n$. Note that if char $R = 2$, then R is commutative if and only if R is anticommutative.

In 1957, Herstein [1] prove

Theorem A. *If for each $x, y \in R$, there exists a natural number $n(x,y) > 1$, depending on x and y, with $[x,y]^{n(x,y)} = [x,y]$, then R is commutative.*

In 1987, Machale [3] proved

Theorem B. *If for each $x, y \in R$, there exists an even natural number $m(x,y)$, depending on x and y, with $(x \circ y)^{m(x,y)} = (x \circ y)$, then R is anticommutative.*

Later in 1987, Yen [4] combined Theorem A and Theorem B to obtain

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Theorem C. *Suppose that for each $x, y \in R$, there exists either a natural number $n(x,y) > 1$ such that $[x,y]^{n(x,y)} = [x,y]$ or an even natural number $m(x,y)$ such that $(x \circ y)^{m(x,y)} = (x \circ y)$. Then for each $x, y \in R$,*
(1) $[x,y] = 0$, if $[x,y]$ is periodic;
(2) $x \circ y = 0$ if both $(x \circ y)$ and $((-y) \circ x)$ are strongly periodic.
Furthermore, R is either commutative or anticommutative; in particular, R is commutative if R is semiprime.

The purpose of this note is to generalize Theorem B and partially extend Theorem A, and combine these generalizations.

2. Results

Lemma 1. *If for each $x, y \in R$, there exist an even natural number $m(x,y)$ and an odd natural number $n(x,y)$, with $[x,y]^{m(x,y)} = [x,y]^{n(x,y)}$, then $[x,y]$ is nilpotent for all $x, y \in R$. Moreover, R is commutative if it has no nonzero nil right ideals.*

Proof. If R is a division ring, then by Herstein's theorem [1, or 2, Theorem 3.1.3], R is commutative.

If R is a primitive ring, then the first possibility is that $R \cong D$, where D is a division ring, and so R is a field. Otherwise, for some $k > 1$, $D_{k'}$ the complete matrix ring over a division ring D, would be a homomorphic image of a subring of R. Thus D_k would inherit the property $[x,y]^{m(x,y)} = [x,y]^{n(x,y)}$, where $m(x,y)$ and $n(x,y)$ are as above. Let e_{ij}'s be the matrix units of D_k. In $D_{k'}$ let $x = e_{11} + e_{12}$ and $y = -e_{21}$. Then $[x,y] = e_{21} + e_{22} - e_{11}$. Hence

$$[x,y]^m = e_{11} + e_{22} \qquad \text{if } m \text{ is an even positive integer, and}$$
$$[x,y]^n = e_{21} + e_{22} - e_{11} \qquad \text{if } n \text{ is an odd positive integer.}$$

This yields a contradiction. Therefore, R is a field.

If R is a semisimple ring, then R is isomorphic to a subdirect sum of primitive rings R_i, each of which, as a homomorphic image of R, satisfies the hypotheses placed on R. Thus each R_i is a field by the result above, and we conclude that R is commutative.

Now, let R be any ring satisfying the hypotheses. Since $R/J(R)$ is a semisimple ring, by the previous result $R/J(R)$ is commutative. Hence, $[x,y] \in J(R)$ for all x, y in R. Now $J(R)$ has the property that if $ab = a$, with $b \in J(R)$, then $a = 0$. Therefore, $[x,y]^{m(x,y)} = [x,y]^{n(x,y)}$ implies that $[x,y]^{n(x,y)} = 0$ for all x, y in R.

Finally, let R be a ring having no nonzero nil right ideals. Since R has no nonzero nil right ideals, R is semiprime. To prove that R is commutative, it suffices to show that if $a \in R$ and $a^2 = 0$ then $a = 0$. For every $x \in R$, we have $0 = [a,x]^{n(a,x)} = [a,x]^{n(a,x)}a$. Thus, $(ax)^{n(a,x)+1} = 0$. So, aR is a nil right ideal. Hence, $aR = 0$. By semiprimeness of R, $aR = 0$ implies that $a = 0$.

Lemma 2. *If for each $x, y \in R$, there exist an even natural number $m(x,y)$ and an odd natural number $n(x,y)$, with $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$, then $(x \circ y)$ is nilpotent for* *all $x, y \in R$. Moreover, R is anticommutative and char $R = 2$ if it has no nonzero nil right ideals.*

Proof. If R is a division ring, then R is anticommutative by Machale's theorem [3].

If R is a primitive ring, then the first possibility is that $R \cong D$, where D is a division ring, and so R is anticommutative. Otherwise, for some $k > 1$, $D_{k'}$ the complete matrix ring over a division ring D, would be a homomorphic image of a subring of R. Thus D_k would inherit the property $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$, where $m(x, y)$ and $n(x, y)$ are as above. Let e_{ij}'s be the matrix units of D_k. In $D_{k'}$ let $x = e_{11} + e_{12}$ and $y = -e_{21}$. Then $(x \circ y) = -e_{11} - e_{21} - e_{22}$. Hence $(x \circ y)^m = e_{11} + me_{21} + e_{22}$ if m is an even positive integer, and $(x \circ y)^n = -e_{11} - ne_{21} - e_{22}$ if n is an odd positive integer. Thus, the hypotheses imply that char $D = 2$. This yields a contradiction. So, R is anticommutative.

If R is a semisimple ring, then R is isomorphic to a subdirect sum of primitive rings R_i, each of which, as a homomorphic image of R, satisfies the hypotheses placed on R. Thus each R_i is anticommutative by the result above, and we conclude that R is anticommutative.

Now, let R be any ring satisfying the hypotheses. Since $R/J(R)$ is a semisimple ring, by the previous result $R/J(R)$ is anticommutative. Hence, $(x \circ y) \in J(R)$ for all x, y in R. Therefore, $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$ implies that $(x \circ y)^{n(x,y)} = 0$ for all x, y in R.

Finally, let R be a ring having no nonzero nil right ideals. Then as the proof of Lemma 1, R is anticommutative. Hence, $2x^2 = (x \circ x) = 0$ for all x in R. Thus, $2R$ is a nil right ideal and so $2R = 0$.

By the proofs of Lemma 1 and Lemma 2, we have the following

Lemma 3. *Suppose that for each $x, y \in R$, there exist an even natural number $m(x, y)$ and an odd natural number $n(x, y)$ such that either*

$$[x, y]^{m(x,y)} = [x, y]^{n(x,y)}$$

or

$$(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}.$$

If R is a semisimple ring, then R is commutative.

Proof. If R is a division ring, then by Lemma 3 of [4], R is commutative.

If R is a primitive ring, then as the proofs of Lemma 1 and Lemma 2, we can show that R is commutative.

If R is a semisimple ring, then as the proof of Lemma 1, we can prove that R is commutative.

Theorem. *Suppose that for each $x, y \in R$, there exist an even natural number $m(x, y)$ and an odd natural number $n(x, y)$ such that either $[x, y]^{m(x,y)} = [x, y]^{n(x,y)}$ or $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$. Then for each $x, y \in R$,*

(1) $[x, y]$ is nilpotent, if $[x, y]$ is $(*)$-periodic;

(2) $(x \circ y)$ *is nilpotent if both* $(x \circ y)$ *and* $((-y) \circ x)$ *are* $(*)$*-periodic. Furthermore,* R *is commutative if* R *has no nonzero nil right ideals.*

Proof. If R is a semisimple ring, then by Lemma 3, R is isomorphic to a subdirect sum of fields.

Now, let R be any ring satisfying the hypotheses. Let $x, y \in R$. Since $[x,y] = [-y,x]$, it suffices to consider the following two cases:

Case 1. $[x,y]$ is $(*)$-periodic. Since $R/J(R)$ is a semisimple ring, by Lemma 3, $R/J(R)$ is commutative. Thus, $[x,y] \in J(R)$. Hence, $[x,y]^{m(x,y)} = [x,y]^{n(x,y)}$ implies that $[x,y]^{n(x,y)} = 0$.

Case 2. $(x \circ y)$ and $((-y) \circ x)$ are $(*)$-periodic. Consider the semisimple ring $\overline{R} = R/J(R)$. By the result above, $\overline{R}$ is isomorphic to a subdirect sum of fields R_i, i in some index set $I \neq \phi$. Let $\Pi_{j \in I} R_j$ be the complete direct sum, and $\phi : \overline{R} \rightarrow \Pi_{j \in I} R_j$ be a monomorphism. Let Π_i be the projection of $\Pi_{j \in I} R_j$ onto R_i, then by definition, $\overline{R}\phi\Pi_i = R_i$ for each $i \in I$. Thus, $(x \circ y)$ is $(*)$-periodic implying that $(\overline{x}\phi\Pi_i \circ \overline{y}\phi\Pi_i)$ is $(*)$-periodic for each $i \in I$. Since R_i is a field, it follows that $(\overline{x}\phi\Pi_i \circ \overline{y}\phi\Pi_i)$ is strongly periodic for each $i \in I$. Similarly, $((-\overline{y}\phi\Pi_i) \circ \overline{x}\phi\Pi_i)$ is strongly periodic for each $i \in I$. By Theorem C, $(\overline{x} \circ \overline{y})\phi\Pi_i = (\overline{x}\phi\Pi_i \circ \overline{y}\phi\Pi_i) = 0$ for each $i \in I$. So, $(\overline{x} \circ \overline{y})\phi = 0$. Since ϕ is one to one, $(\overline{x} \circ \overline{y}) = 0$, i.e., $(x \circ y) \in J(R)$. Hence, $(x \circ y)^{m(x,y)} = (x \circ y)^{n(x,y)}$ implies that $(x \circ y)^{n(x,y)} = 0$.

Finally, let R have no nonzero nil right ideals. Then as the proofs of Lemma 1 and Lemma 2, we have that either $[x,y] = 0$ or $(x \circ y) = 0$ for all x, y in R. Thus, R is commutative by the result of [4].

Finally, we note that Lemma 1 does not hold if both of $m(x,y)$ and $n(x,y)$ are even or odd. To see this, let F be a finite field. Observing that $[x,y]^2$ is a scalar matrix for all x, y in F_2, it is easy to see that the hypotheses as in Lemma 1 hold. However, $[x,y]$ is not necessarily nilpotent for all x, y in F_2. We also note that Lemma 2 does not hold if both of $m(x,y)$ and $n(x,y)$ are even or odd.

References

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