TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 3, Autumn 1991

A REMARK ON BOUNDEDNESS OF THE NUMERICAL RANGE

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Abstract. It is shown that Chien's result on the smallest rectangle with sides parallel to the axes containing the numerical range of a real matrix follows from a simple observation on complex matrices. A counter-example to Chien's conjecture is also given.

The numerical range of an m-square complex matrix B is defined to be

$$W(B) = \{(Bx,x) : \| x \| = 1, \quad x \in \mathbf{C}^m\},$$

where $(.,.)$ denotes the standard inner product in $\mathbf{C}^m$. Let $\Lambda(B)$ denote the set of eigenvalues of B. If B is hermitian, let

$$\lambda_1(B) = \max\Lambda(B) \qquad \text{and} \qquad \lambda_m(B) = \min\Lambda(B).$$

Let $A = [a_{ij}]$ be an n-square complex matrix. Then, for $x \in C^n$,

$$(Ax,x) = \frac{1}{2}((A + A^*)x, x) + \frac{i}{2}(i(A^* - A)x, x).$$

Since $A + A^*$ and $i(A^* - A)$ are hermitian, it follows that

$$W(A + A^*) = [\lambda_n(A + A^*), \lambda_1(A + A^*)],$$

and

$$W(i(A^* - A)) = [\lambda_n(i(A^* - A), \lambda_1(i(A^* - A))].$$

Hence $[a,b] \times [c,d]$ is the smallest rectangle with sides parallel to the axes containing $W(A)$, where

$$a = \frac{1}{2}\lambda_n(A + A^*),$$

$$b = \frac{1}{2}\lambda_1(A + A^*),$$

$$c = \frac{1}{2}\lambda_n(i(A^* - A)) = \frac{1}{2}\min\{Im(\lambda) : \lambda \in \Lambda(A - A^*)\},$$

and

$$d = \frac{1}{2}\lambda_1(i(A^* - A)) = \frac{1}{2}\max\{Im(\lambda) : \lambda \in \Lambda(A - A^*)\}.$$

Received October 12, 1990.

Suppose now that A is real. M. T. Chien [1] proved that

$$c = \frac{1}{2}\lambda_n(K) \qquad \text{and} \qquad d = \frac{1}{2}\lambda_1(K),$$

where

$$K = \begin{bmatrix} 0 & A - A^T \\ A^T - A & 0 \end{bmatrix}.$$

We shall see that a shorter proof is obtained from the previous observation. It is obvious (see, e.g., [2, pp. 45, 46]) that

$$\Lambda\left(\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}\right) = \{\alpha \in \mathbb{R} : \alpha^2 \in \Lambda(BB^*)\}.$$

As $A - A^T$ is skew-symmetric,

$$\begin{aligned} \Lambda(K) &= \{\alpha \in \mathbb{R} : \alpha = \pm Im(\lambda), \quad \lambda \in \Lambda(A - A^T)\} \\ &= \{\alpha \in \mathbb{R} : \alpha = Im(\lambda), \quad \lambda \in \Lambda(A - A^T)\}, \end{aligned}$$

and Chien's result follows. Note also that $\lambda_n(K) = -\lambda_1(K)$, so $c = -d$.

We study some particular cases:

(i) $n = 2$. $\Lambda(A - A^T) = \{\lambda i, -\lambda i\}$, where $\lambda \geq 0$, and the eigenvalues of K are $\pm\lambda$. Since

$$\begin{aligned} 2\lambda^2 &= \operatorname{tr}((A - A^T)^T(A - A^T)) = \sum_{i,j}(a_{ij} - a_{ji})^2, \\ \lambda^2 &= \sum_{i<j}(a_{ij} - a_{ji})^2. \end{aligned} \tag{1}$$

(ii) $n = 3$. $\Lambda(A - A^T) = \{\lambda i, -\lambda i, 0\}$, where $\lambda \geq 0$, and the eigenvalues of K are $0, \pm\lambda$. Similarly, we have (1).

Chien [1] conjectured for any n that if λ is an eigenvalue of K, then $\lambda = 0$ or λ satisfies (1).

(iii) $n = 4$. $\Lambda(A - A^T) = \{\lambda i, -\lambda i, \mu i, -\mu i\}$, where $\lambda, \mu \geq 0$, and the eigenvalues of K are $\pm\lambda$, $\pm\mu$. Similarly,

$$\lambda^2 + \mu^2 = \sum_{i<j}(a_{ij} - a_{ji})^2.$$

Let $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\Lambda(B - B^T) = \{i, -i\}$. Let $A = B \oplus B$. Then the eigenvalues of K are ± 1, whereas $\sum_{i<j}(a_{ij} - a_{ji})^2 = 2$. Chien's conjecture fails.

Acknowledgement

The author is grateful to Dr. Yik-Hoi Au-Yeung for his guidance and encouragement and to Mr. Che-Man Cheng for his useful comment.

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