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APPLICATION OF THE METHOD OF THE TWO-SIDED APPROXIMATIONS TO THE SOLUTION OF THE PERIODIC PROBLEM FOR IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In the paper the application of the method of the two-sided approximations to finding the periodic solutions of impulsive differential equations is justified.

1. Introduction

Consider the impulsive T-periodic differential equation

$$\begin{aligned} \frac{dx}{dt} &= f(t,x), \qquad t \neq \tau_k, \\ \Delta x &= I_k(x), \qquad t = \tau_k, \end{aligned} \tag{1}$$

where $t \in \mathbb{R} = (-\infty, \infty)$, $k \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, $x = col(x_1, \ldots, x_n) \in D \subset \Re^n$ and $\Re^n$ is an n-dimensional vector space with norm $\|x\| = \max_{1 \leq i \leq n} |x_i|$.

Impulsive differential equations of the form (1) are an object of active research in the recent years [1]-[10]. We shall note that the solution $x(t)$ of (1) for $t \neq \tau_k$ satisfies the differential equation $dx/dt = f(t,x)$ and for $t = \tau_k$ it satisfies the condition for a jump $\Delta x = I_k(x)$ and $x(\tau_k^+) = x(\tau_k) + I_k(x(\tau_k))$, $x(\tau_k^-) = x(\tau_k)$. Here $x(\tau_k^\pm) = lim_{t \to \tau_k \pm 0} x(t)$.

To find the T periodic solutions of equation (1) we shall apply the method of the two-sided approximations [11].

2. Preliminary Notes

Let the functions $g(t,x,y)$ and $J_k(x,y)$ be such that

$$g(t,x,x) = f(t,x), \; J_k(x,x) = I_k(x) \qquad (t \in \mathbb{R}, \; k \in \mathbb{Z}, \; x \in D) \tag{2}$$

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and instead of (1) consider the equation

$$\begin{aligned} \frac{dx}{dt} &= g(t,x,x), \qquad t \neq \tau_k, \\ \Delta x &= J_k(x,x), \qquad t = \tau_k. \end{aligned} \tag{3}$$

Assume that the following conditions (H) hold:

H1. $\tau_0 = 0$, $\tau_k < \tau_{k+1}$ $(k \in \mathbb{Z})$ and there exists an integer $q > 0$ such that $\tau_{k+q} = \tau_k + T$ $(k \in \mathbb{Z})$.

H2. The function $g : \mathbb{R} \times D \times D \to \mathbb{R}^n$ is continuous in the sets $(\tau_{k-1}, \tau_k] \times D \times D$ $(k \in \mathbb{Z})$ and for any $x, y \in D$ and $k \in \mathbb{Z}$ there exists the finite limit of $g(t,u,v)$ as $(t,u,v) \to (\tau_k, x, y), t > \tau_k$.

H3. The functions $J_k : D \times D \to \mathbb{R}^n (k = 1, \cdots, q)$ are continous in $D \times D$.

H4. $g(t+T, x, y) = g(t,x,y)$ and $J_{k+q}(x,y) = J_k(x,y)$ for $t \in \mathbb{R}$, $k \in \mathbb{Z}$; $x, y \in D$.

H5. There exist $M, \mu, L_k, \ell_k \in \mathbb{R}^n$ such that the following inequalities hold

$$\mu \leq g(t,x,y) \leq M, \qquad \ell_k \leq J_k(x,y) \leq L_k, \tag{4}$$

$$g(t,x,y) \leq g(t,u,v), \qquad J_k(x,y) \leq J_k(u,v) \tag{5}$$

for $t \in \mathbb{R}$; $k \in \mathbb{Z}$; $x, y, u, v \in \mathbb{R}^n$, $x \leq u$, $v \leq y$, where $x \leq u$ means that $x_i \leq u_i$ $(i = 1, \cdots, n)$.

H6. $D = \{x \in \mathbb{R}^n : a \leq x \leq b\}$ and $b - a > \frac{T}{2}(M - \mu) + 2\sum_{k=1}^{q} max(|L_k|, |\ell_k|) = 2\epsilon$ where $|x| = col(|x_1|, \ldots, |x_n|)$, $max(x,y) = col(max(x_1,y_1), \ldots, max(x_n,y_n))$.

Remark 1. We shall note that if N is a nonnegative $(n \times n)$-matrix and $-N \leq \frac{\partial f}{\partial x}(t,x) \leq N$ then the function $g(t,x,y) = \frac{1}{2}[f(t,x) + Nx] + \frac{1}{2}[f(t,y) - Ny]$ satisfies conditions (2) and (5).

If the function $h(t,x)$ is such that

$$h(t,x) - h(t,u) \leq f(t,x) - f(t,u) \leq -h(t,x) + h(t,u)$$

for $x \leq u$, then the function

$$g(t,x,y) = \frac{1}{2}[f(t,x) + h(t,x)] + \frac{1}{2}[f(t,y) - h(t,y)]$$

also satisfies conditions (2) and (5).

Let $x_0 \in D_\epsilon = \{x \in \mathbb{R}^n : a + \epsilon \leq x \leq b - \epsilon\}$ and define successively the sequences $\{u_m(t,x_0)\}$ and $\{v_m(t,x_0)\}$ of T-periodic functions which in the interval $[0,T]$

are given by the formulae:

$$u_0(t,x_0) = x_0 - \frac{M-\mu}{2}\alpha(t) + (1-\frac{t}{T})\sum_{0\le\tau_k<t}\ell_k - \frac{t}{T}\sum_{t\le\tau_k<T}L_k, \tag{6}$$

$$v_0(t,x_0) = x_0 + \frac{M-\mu}{2}\alpha(t) + (1-\frac{t}{T})\sum_{0\le\tau_k<t}L_k - \frac{t}{T}\sum_{t\le\tau_k<T}\ell_k, \tag{7}$$

$$\begin{aligned} u_{m+1}(t,x_0) &= x_0 + (1-\frac{t}{T})\int_0^t g(s,u_m(s,x_0),v_m(s,x_0))ds \\ &- \frac{t}{T}\int_t^T g(s,v_m(s,x_0),u_m(s,x_0))ds + (1-\frac{t}{T})\sum_{0\le\tau_k<t}J_k(u_m(\tau_k,x_0),v_m(\tau_k,x_0)) \\ &- \frac{t}{T}\sum_{t\le\tau_k<T}J_k(v_m(\tau_k,x_0),u_m(\tau_k,x_0)), \end{aligned} \tag{8}$$

$$\begin{aligned} v_{m+1}(t,x_0) &= x_0 + (1-\frac{t}{T})\int_0^t g(s,v_m(s,x_0),u_m(s,x_0))ds \\ &- \frac{t}{T}\int_t^T g(s,u_m(s,x_0),v_m(s,x_0))ds + (1-\frac{t}{T})\sum_{0\le\tau_k<t}J_k(v_m(\tau_k,x_0),u_m(\tau_k,x_0)) \\ &- \frac{t}{T}\sum_{t\le\tau_k<T}J_k(u_m(\tau_k,x_0),v_m(\tau_k,x_0)), \end{aligned} \tag{9}$$

where $\alpha(t) = 2t(1-\frac{t}{T})$ for $t \in [0,T]$.

We shall find sufficient conditions under which the sequences $u_m(t,x_0)$ and $v_m(t,x_0)$ two-sided and monotonely tend to the T-periodic solution $\tilde{x}(t,x_0)$ of equation (3) for which $\tilde{x}(0,x_0) = x_0$.

In the proof of the main results we shall use the following relations which are valid for $t \in [0,T]$:

$$(1-\frac{t}{T})\int_0^t ds + \frac{t}{T}\int_t^T ds = \alpha(t) \le \frac{T}{2}, \tag{10}$$

$$(1-\frac{t}{T})\int_0^t \alpha(s)\,ds + \frac{t}{T}\int_t^T \alpha(s)\,ds = \frac{\alpha^2(t)}{3} + \frac{T\alpha(t)}{2} \le \frac{T}{3}\alpha(t), \tag{11}$$

$$(1-\frac{t}{T})i[0,t) + \frac{t}{T}i[t,T) \le q[1-(q-1)\frac{\theta}{T}] \equiv Q, \tag{12}$$

$$(1-\frac{t}{T})\sum_{0\le\tau_k<t}\alpha(\tau_k) + \frac{t}{T}\sum_{t\le\tau_k<T}\alpha(\tau_k) \le \frac{8Tq}{27} \equiv S, \tag{13}$$

$$(1-\frac{t}{T})\sum_{0\le\tau_k<t}\gamma_k + \frac{t}{T}\sum_{t\le\tau_k<T}\gamma_k \le \sum_{k=1}^{q}\gamma_k, \tag{14}$$

where $\gamma_k \in \mathbb{R}^n$, $\gamma_k \ge 0$, $i[t,s)$ is the number of point $\{\tau_k\}$ lying in the interval $[t,s)$ and $\theta = min_{k=1,\cdots,q}(\tau_k - \tau_{k-1})$.

3. Main Results

Theorem 1. *Let conditions (H) and $x_0 \in D_\epsilon$.*
Then:
1) The functions $u_m(t, x_0)$, $v_m(t, x_0)$ satisfy the relations:

$$u_m(0, x_0) = u_m(T, x_0) = v_m(0, x_0) = v_m(T, x_0) = x_0, \tag{15}$$

$$u_0(t, x_0) \le u_1(t, x_0) \le \dots \le u_m(t, x_0), \tag{16}$$

$$v_0(t, x_0) \ge v_1(t, x_0) \ge \dots \ge v_m(t, x_0), \tag{17}$$

$$a \le u_m(t, x_0) \le v_m(t, x_0) \le b \tag{18}$$

for $t \in [0, T]$, $m = 0, 1, 2, \cdots$.

2) The sequences $\{u_m(t, x_0)\}$, $\{v_m(t, x_0)\}$ are uniformly convergent in the interval $[0, T]$ and their limits $u(t, x_0)$, $v(t, x_0)$ satisfy the relations:

$$u(0, x_0) = u(T, x_0) = v(0, x_0) = v(T, x_0) = x_0, \tag{19}$$

$$u_m(t, x_0) \le u(t, x_0) \le v(t, x_0) \le v_m(t, x_0) \; (t \in [0, T],\ m = 0, 1, 2, \cdots), \tag{20}$$

$$\begin{aligned} u(t, x_0) &= x_0 + (1 - \frac{t}{T}) \int_0^t g(s, u(s, x_0), v(s, x_0)) ds \\ &- \frac{t}{T} \int_t^T g(s, v(s, x_0), u(s, x_0)) ds + (1 - \frac{t}{T}) \sum_{0 \le \tau_k < t} J_k(u(\tau_k, x_0), v(\tau_k, x_0)) \\ &- \frac{t}{T} \sum_{t \le \tau_k < T} J_k(v(\tau_k, x_0), u(\tau_k, x_0)), \end{aligned} \tag{21}$$

$$\begin{aligned} v(t, x_0) &= x_0 + (1 - \frac{t}{T}) \int_0^t g(s, v(s, x_0), u(s, x_0)) ds \\ &- \frac{t}{T} \int_t^T g(s, u(s, x_0), v(s, x_0)) ds + (1 - \frac{t}{T}) \sum_{0 \le \tau_k < t} J_k(v(\tau_k, x_0), u(\tau_k, x_0)) \\ &- \frac{t}{T} \sum_{t \le \tau_k < T} J_k(u(\tau_k, x_0), v(\tau_k, x_0)). \end{aligned} \tag{22}$$

Proof.

1) The validity of (15) is obvious. From (6), (7), H6 and property (14) it follows that

$$a \le u_0(t, x_0) \le v_0(t, x_0) \le b. \tag{23}$$

Taking into account (4) and (9), we obtain that

$$v_1(t, x_0) \le x_0 + (1 - \frac{t}{T}) \int_0^t M ds - \frac{t}{T} \int_t^T \mu ds + (1 - \frac{t}{T}) \sum_{0 \le \tau_k < t} L_k - \frac{t}{T} \sum_{t \le \tau_k < T} \ell_k = v_0(t, x)$$

for $t \in [0,T]$. Analogously, $u_0(t,x_0) \leq u_1(t,x_0)$ for $t \in [0,T]$.

From (8), (9), (23) and condition (5) it follows that

$$\begin{aligned}
&v_1(t,x_0) - u_1(t,x_0) \\
&= (1-\frac{t}{T})\int_0^t [g(s,v_0(s,x_0),u_0(s,x_0)) - g(s,u_0(s,x_0),v_0(s,x_0))]\,ds \\
&+ \frac{t}{T}\int_t^T [g(s,v_0(s,x_0),u_0(s,x_0)) - g(s,u_0(s,x_0),v_0(s,x_0))]\,ds \\
&+ (1-\frac{t}{T})\sum_{0\leq\tau_k<t} [J_k(v_0(\tau_k,x_0),u_0(\tau_k,x_0)) - J_k(u_0(\tau_k,x_0),v_0(\tau_k,x_0))] \\
&+ \frac{t}{T}\sum_{t\leq\tau_k<T} [J_k(v_0(\tau_k,x_0),u_0(\tau_k,x_0)) - J_k(u_0(\tau_k,x_0),v_0(\tau_k,x_0))]
\end{aligned}$$

Thus for $t \in [0,T]$ we have

$$a \leq u_0(t,x_0) \leq u_1(t,x_0) \leq v_1(t,x_0) \leq v_0(t,x_0) \leq b. \tag{24}$$

By induction on m in virtue of (24) it is proved that for any $m = 0,1,2,\cdots$ and $t \in [0,T]$

$$u_{m+1}(t,x_0) \leq u_m(t,x_0) \leq v_m(t,x_0) \leq v_{m+1}(t,x_0).$$

2) Consider the space $PC(\mathbb{R},\mathbb{R}^n)$ of piecewise continuous functions $x : \mathbb{R} \to \mathbb{R}^n$ which have points of discontinuity τ_k $(k \in \mathbb{Z})$ and are continuous from the left in $\mathbb{R}$. Let the norm of $x \in PC(\mathbb{R},\mathbb{R}^n)$ be $\|x\|_{PC} = \sup_{t\in\mathbb{R}} \|x(t)\|$.

From conditions **H2** and **H3** it follows that the functions $u_m(t,x_0)$ and $v_m(t,x_0)$ belong to $PC(\mathbb{R},\mathbb{R}^n)$. Since the sequences $\{u_m(t,x_0)\}$ and $\{v_m(t,x_0)\}$ are uniformly bounded and quasiequicontinuous [5], then by Lemma 4, [5] they have convergent subsequences. But from the monotonicity of the sequences $\{u_m(t,x_0)\}$ and $\{v_m(t,x_0)\}$ it follows that $u_m(t,x_0)$ and $v_m(t,x_0)$ are convergent in $PC(\mathbb{R},\mathbb{R}^n)$ which means that there exist functions $u(t,x_0)$ and $v(t,x_0)$ of $PC(\mathbb{R},\mathbb{R}^n)$ for which

$$\lim_{m\to\infty} u_m(t,x_0) = u(t,x_0), \qquad \lim_{m\to\infty} v_m(t,x_0) = v(t,x_0) \tag{25}$$

uniformly with respect to $t \in [0,T]$.

(15)-(18) and (25) imply immediately the validity of relations (19)-(22). Theorem 1 is proved.

Consider the equation

$$\begin{aligned}
x(t) &= x_0 + (1-\frac{t}{T})\int_0^t g(s,x(s),x(s))\,ds - \frac{t}{T}\int_t^T g(s,x(s),x(s))\,ds \\
&+ (1-\frac{t}{T})\sum_{0\leq\tau_k<t} J_k(x(\tau_k),x(\tau_k)) - \frac{t}{T}\sum_{t\leq\tau_k<T} J_k(x(\tau_k),x(\tau_k)).
\end{aligned} \tag{26}$$

Theorem 2. *Let conditions (H) hold and* $x_0 \in D_\epsilon$.
Then equation (26) has a T-periodic solution $x^*(t)$ *and the following relations hold:*

$$\begin{aligned} &x^*(0) = x^*(T) = x_0, \\ &u_m(t, x_0) \le x^*(t) \le v_m(t, x_0) \qquad (t \in [0,T],\ m = 0,1,2,\cdots), && (27) \\ &u(t, x_0) \le x^*(t) \le v(t, x_0) \qquad (t \in [0,T]). && (28) \end{aligned}$$

Proof. Consider the set Ω of functions $x \in PC(\mathbb{R}, \mathbb{R}^n)$ which are T-periodic and satisfy the conditions

$$x(0) = x(T) = x_0, \qquad a \le x(t) \le b.$$

Define the operator $\mathcal{F} : \Omega \to PC(\mathbb{R}, \mathbb{R}^n)$ by the formula

$$\begin{aligned} \mathcal{F}_{x(t)} = x_0 &+ (1 - \frac{t}{T}) \int_0^t g(s, x(s), x(s))\, ds - \frac{t}{T} \int_t^T g(s, x(s), x(s))\, ds \\ &+ (1 - \frac{t}{T}) \sum_{0 \le \tau_k < t} J_k(x(\tau_k), x(\tau_k)) - \frac{t}{T} \sum_{t \le \tau_k < T} J_k(x(\tau_k), x(\tau_k)). \end{aligned}$$

The following assertions are valid:
I. The set Ω is bounded, convex and closed in $PC(\mathbb{R}, \mathbb{R}^n)$.
II. $\mathcal{F}$ maps Ω into itself. Indeed, if $x \in \Omega$, then

$$\begin{aligned} \mathcal{F}x(t) \le x_0 &+ (1 - \frac{t}{T}) \int_0^t M\, ds - \frac{t}{T} \int_t^T \mu\, ds \\ &+ (1 - \frac{t}{T}) \sum_{0 \le \tau_k < t} L_k - \frac{t}{T} \sum_{t \le \tau_k < t} \ell_k = v_0(t, x_0) \le b. \qquad (29) \end{aligned}$$

Analogously,

$$\mathcal{F}x(t) \ge u_0(t, x_0) \ge a \qquad (30)$$

and since $\mathcal{F}x(0) = \mathcal{F}x(T) = x_0$ and $\mathcal{F}x \in PC(\mathbb{R}, \mathbb{R}^n)$, then $\mathcal{F}x \in \Omega$.

III. The set $\mathcal{F}\Omega$ is relatively compact in $PC(\mathbb{R}, \mathbb{R}^n)$. For the proof of this assertion we apply Lemma 4, [5], taking into account that $\mathcal{F}\Omega$ is uniformly bounded and quasiequicontinuous. We shall mention only that the quasiequicontinuity of $\mathcal{F}\Omega$ follows from H2, H3 and the equality

$$\begin{aligned} \mathcal{F}x(t_2) - \mathcal{F}x(t_1) = &\int_{t_1}^{t_2} g(s, x(s), x(s))\, ds \\ &+ \frac{t_1 - t_2}{T} [\int_0^T g(s, x(s), x(s))\, ds + \sum_{k=1}^{q} J_k(x(\tau_k), x(\tau_k))] \end{aligned}$$

for $x \in \Omega$ and $t_1, t_2 \in (\tau_{k-1}, \tau_k]$, $t_1 < t_2\,(k \in \mathbb{Z})$.

Hence by the Schauder-Tychonoff theorem the operator $\mathcal{F}$ has a fixed point $x^* \in \Omega$, i.e. there exists a T-periodic function $x^*(t)$ satisfying (26). From (29) and (30) there follows the estimate

$$u_0(t, x_0) \leq x^*(t) \leq v_0(t, x_0)$$

from which by induction on $m = 0, 1, 2, \cdots$ we obtain that

$$u_m(t, x_0) \leq x^*(t) \leq v_m(t, x_0) \qquad (t \in [0, T],\ m = 0, 1, 2, \cdots). \tag{31}$$

In (31) we pass to the limit and obtain (28).

Theorem 3. *Let the following conditions be fulfilled:*

1) Conditions (H) hold and $x_0 \in D_\epsilon$.

2) The functions g and J_k satisfy the estimates

$$g(t, x, y) - g(t, y, x) \leq K(x - y),$$
$$J_k(x, y) - J_k(y, x) \leq C(x - y),$$

where $a \leq y \leq x \leq b$ and K and C are nonnegative $(n \times n)$-matrices.

3) The modules of the eigenvalues of the matrix

$$P = \begin{bmatrix} \frac{T}{3}K & K \\ SC & QC \end{bmatrix} \tag{32}$$

are less than 1.

Then equation (26) has a unique T-periodic solution $\tilde{x}(t, x_0)$ for which $\tilde{x}(0, x_0) = x_0$ and

$$\tilde{x}(t, x_0) = u(t, x_0) = v(t, x_0) \qquad (t \in [0, T]).$$

Proof. For $m = 0$ the following estimate is valid

$$v_0(t, x_0) - u_0(t, x_0) = \alpha(t)(M - \mu) + (1 - \frac{t}{T}) \sum_{0 \leq \tau_k < t} (L_k - \ell_k) + \frac{t}{T} \sum_{t \leq \tau_k < T} (L_k - \ell_k)$$
$$\leq \alpha(t)(M - \mu) + \sum_{k=1}^{q} (L_k - \ell_k) = \alpha(t)a_0 + b_0.$$

For $m = j$ let us have that

$$v_j(t, x_0) - u_j(t, x_0) \leq \alpha(t)a_j + b_j.$$

Then by (8), (9), (32), (10)-(13) we obtain

$$\begin{aligned} & v_{j+1}(t,x_0) - u_{j+1}(t,x_0) \\ & \leq (1-\frac{t}{T})\int_0^t K[\alpha(s)a_j+b_j]ds + \frac{t}{T}\int_t^T K[\alpha(s)a_j+b_j]ds \\ & \quad + (1-\frac{t}{T})\sum_{0\leq\tau_k<t} C[\alpha(\tau_k)a_j+b_j] + \frac{t}{T}\sum_{t\leq\tau_k<T} C[\alpha(\tau_k)a_j+b_j] \\ & \leq \alpha(t)\big[\frac{T}{3}Ka_j + Kb_j\big] + \big[SCa_j + QCb_j\big]. \end{aligned}$$

Consequently, for $m = 0, 1, 2, \cdots$ the following estimates are valid

$$v_m(t,x_0) - u_m(t,x_0) \leq \alpha(t)a_m + b_m \qquad (t \in [0,T])$$

where

$$\begin{aligned} a_{m+1} &= \frac{T}{3}Ka_m + Kb_m, & a_0 &= M - \mu, \\ b_{m+1} &= SCa_m + QCb_m, & b_0 &= \sum_{k=1}^{q}(L_k - \ell_k). \end{aligned} \tag{33}$$

From (3), in view of condition 3 it follows that $a_m \to 0$ and $b_m \to 0$ as $m \to \infty$, i.e.

$$lim_{m\to\infty}[v_m(t,x_0) - u_m(t,x_0)] = 0 \qquad \text{(uniformly with respect to } t \in [0,T]).$$

Then $u(t,x_0) = v(t,x_0)$ and by Theorem 2

$$\tilde{x}(t,x_0) = x^*(t) = u(t,x_0) = v(t,x_0) \qquad (t \in [0,T]).$$

Theorem 3 is proved.

Under the conditions of Theorem 3 we shall consider the question of existence of a T-periodic solution of equation (3).

Introduce the mapping $\Delta(x_0) : D_\epsilon \to \mathbb{R}^n$:

$$\Delta(x_0) = \frac{1}{T}\int_0^T g(s,\tilde{x}(s,x_0),\tilde{x}(s,x_0))ds + \frac{1}{T}\sum_{k=1}^{q} J_k(\tilde{x}(\tau_k,x_0),\tilde{x}(\tau_k,x_0)),$$

where $\tilde{x}(t,x_0)$ is the T-periodic solution of equation (26) for which $\tilde{x}(0,x_0) = x_0$.

Since

$$\begin{aligned} \tilde{x}(t,x_0) &= x_0 + \int_0^t g(s,\tilde{x}(s,x_0),\tilde{x}(s,x_0))\,ds \\ &\quad + \sum_{0\leq\tau_k<t} J_k(\tilde{x}(\tau_k,x_0),\tilde{x}(\tau_k,x_0)) - t\,\Delta\,(x_0) \end{aligned}$$

then $\tilde{x}(t, x_0)$ is a T-periodic solution of (3) if and only if $\Delta(x_0) = 0$.

From condition (5) and (27) it follows that

$$\Delta_m(x_0) \leq \Delta(x_0) \leq \Delta^m(x_0) \qquad (x_0 \in D_\epsilon) \tag{34}$$

where

$$\Delta_m(x_0) = \frac{1}{T}\int_0^T g(s, u_m(s, x_0), v_m(s, x_0))\, ds + \frac{1}{T}\sum_{k=1}^{q} J_k(u_m(\tau_k, x_0), v_m(\tau_k, x_0)),$$

$$\Delta^m(x_0) = \frac{1}{T}\int_0^T g(s, v_m(s, x_0), u_m(s, x_0))\, ds + \frac{1}{T}\sum_{k=1}^{q} J_k(v_m(\tau_k, x_0), u_m(\tau_k, x_0)).$$

Inequalities (34) imply the following theorem.

Theorem 4. *Let the conditions of Theorem 3 hold and for some integer $m \geq 0$, $\Delta_m(x_0) > 0$ or $\Delta^m(x_0) < 0$.*

Then equation (3) has no T-periodic solution $x(t)$ for which $x(0) = x_0$.

The following theorem is also valid.

Theorem 5. *Let the following conditions be fulfilled:*

1) The conditions of Theorem 3 hold.

2) For some integer $m \geq 0$ the mapping $\Delta_m(x_0)$ has an isolated singular point x_0 ($\Delta_m(x_0) = 0$).

3) The index of the singular point x_0 is different from zero.

4) There exists a closed domain $\mathbf{F} \subset D_\epsilon$ with a unique singular point x_0 such that on its boundary $\partial\mathbf{F}$ the inequality

$$\| [\frac{T}{3}K + \frac{1}{T}\sum_{k=1}^{q}\alpha(\tau_k)C]a_m + [K + \frac{q}{T}C]b_m \| \leq \inf_{x\in\partial\mathbf{F}} \| \Delta_m(x) \| \tag{35}$$

holds, where a_m and b_m are defined by formulae (33).

Then equation (3) has a T-periodic solution $x(t)$ for which $x(0) \in \mathbf{F}$.

Proof. Following the proof of Theorem 5. 5, [11], p. 166, it suffices to prove that

$$\| \Delta_m(x) \| > \| \Delta(x) - \Delta_m(x) \| \qquad (x \in \partial\mathbf{F})$$

which follows from (34), (35) and the estimate

$$\begin{aligned} 0 &\leq \Delta(x) - \Delta_m(x) \leq \Delta^m(x) - \Delta_m(x) \\ &\leq \frac{1}{T}\int_0^T K(v_m(s, x_0) - u_m(s, x_0))\, ds + \frac{1}{T}\sum_{k=1}^{q} C(v_m(\tau_k, x_0) - u_m(\tau_k, x_0)) \\ &\leq \frac{1}{T}\int_0^T K[\alpha(s)a_m + b_m]\, ds + \frac{1}{T}\sum_{k=1}^{q} C[\alpha(\tau_k)a_m + b_m] \\ &= [\frac{T}{3}K + \frac{1}{T}\sum_{k=1}^{q}\alpha(\tau_k)C]a_m + [K + \frac{q}{T}C]b_m. \end{aligned}$$

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