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SOME COMMUTATIVITY THEOREMS FOR ASSOCIATIVE RINGS WITH CONSTRAINTS INVOLVING A NIL SUBSET

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Abstract. We first prove that a ring R with unity 1 is commutative if and only if for each x in R either x is central or there exists a polynomial $f(t) \in Z[t]$ such that $x-x^2f(x) \in A$, where A is a nil subset of R (not necessarily a subring of R) and R stisfies any one of the conditions $[x, x^m y - x^p y^n x^q] = 0$ and $[x, yx^m - x^p y^n x^q] = 0$ for all x, y in R, where $m \geq 0$, $n > 1$, $p \geq 0$, $q \geq 0$ are integers depending on pair of elements x, y. Further the same result has been extended for one sided s-unital rings. Finally a related result for a nil commutative subset A is also obtained.

1. Introduction.

Let A be a non-empty subset (not necessarily a subring) of an associative ring R (R may be without unity 1); let N, Z denote the set of nilpotent elements of R, the center of R respectively. A ring R is called right (resp. left) s-unital if $x \in xR$ (resp. $x \in Rx$) for all x in R. R is called s-unital if $x \in xR \cap Rx$ for all x in R. R is called normal if every idempotent of R is central. The symbol $[x, y]$ stands for the commutator $xy - yx$, for any pair of elements of R. As usual $Z[t]$ is the totality of polynomials in t with coefficients in Z, the ring of integers. We consider the following conditions:

(I-A) For each x in R there exists a polynomial $f(t) \in Z[t]$ such that $x - x^2f(x) \in A$.

(II-A) For each x in R either x is central or there exists a polynomial $f(t) \in Z[t]$ such that $x - x^2f(x) \in A$.

(III-A) For each $x \in R$ and $a \in A$, $[[a, x], x] = 0$.

(IV) For each x, y in R there exist integers $m = m(x, y) \geq 0$, $n = n(x, y) > 1$, $p = p(x, y) \geq 0$, $q = q(x, y) \geq 0$ such that $[x, x^m y - x^p y^n x^q] = 0$.

(V) For each x, y in R there exist integers $m = m(x, y) \geq 0$, $n = n(x, y) > 1$, $p = p(x, y) \geq 0$, $q = q(x, y) \geq 0$ such that $[x, yx^m - x^p y^n x^q] = 0$.

A classicial theorem of Herstein [8] establishes commutativity of all rings satisfying (I-Z). Many authors have studied the commutativity of rings satisfying the condition (I-A), but always under some restrictions on A (for a complete reference see [5]). Various

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special cases of (IV) and (V) are shown to imply commutativity of rings (cf. [2], [3]) for example if the integral indices in the underlying conditions are 'global'. The objective of the present paper is to investigate the commutativity of R, when the integral indices are 'local' i.e. they are depending on pair of elements x, y for their values. We establish commutativity of R, when (II-A) holds for a nil subset A together with either of the conditions (IV) and (V). Moreover, the commutativity of ring R satisfying (II-A) for a commutative nil subset A has been studied. The results obtained here, improve the main theorems of the author et. al. [2] and [3].

2. Property (II-A) For Noncommutative A

Theorem 1. *Let R be a ring with unity 1. The following statements are equivalent:*

(i) R is commutative.

(ii) R satisfies either of the conditions (IV) & (V) and there exists a nil subset A of R for which R satisfies $(II-A)$.

For easy reference, we state the following well-known lemma.

Lemma 1 [10]. *Let R be a ring with unity 1 and $f : R \to R$ be a function such that $f(x) = f(x+1)$ holds for all x in R. If for any x in R, there exists a positive integer h such that $x^h f(x) = 0$, then necessarily $f(x) = 0$.*

Proof of Theorem 1. Obviously (i)$\Longrightarrow$(ii). Next, to show that (ii)$\Longrightarrow$(i) suppose that $a \in N$ and x be an arbitrary element of R. If R satisfies (IV), then there exist integers $m_1 \geq 0$, $n_1 > 1$, $p_1 \geq 0$, $q_1 \geq 0$ depending on the pair of elements x and a such that $x^{m_1}[x,a] = x^{p_1}[x,a^{n_1}]x^{q_1}$. Again if we choose $m_2 \geq 0$, $n_2 > 1$, $p_2 \geq 0$, $q_2 \geq 0$ depending on the pair of elements x and a^{n_1}, then $x^{m_2}[x,a^{n_1}] = x^{p_2}[x,(a^{n_1})^{n_2}]x^{q_2}$. Thus for any positive integer k we have integers $m_1, m_2, \ldots, m_k \geq 0$, $n_1, n_2, \ldots, n_k > 1$, $p_1, p_2, \ldots, p_k \geq 0$ and $q_1, q_2, \ldots, q_k \geq 0$ such that

$$\begin{aligned} x^{m_1+m_2+\ldots+m_k}[x,a] &= x^{m_2+\ldots+m_k}\, x^{p_1}[x,a^{n_1}]x^{q_1} \\ &= x^{m_3+\ldots+m_k}\, x^{p_1+p_2}[x,a^{n_1 n_2}]x^{q_1+q_2} \\ &= ----------- \\ &= ----------- \\ &= x^{p_1+p_2+\ldots+p_k}[x,a^{n_1 n_2 \ldots n_k}]x^{q_1+\ldots+q_k}. \end{aligned}$$

Hence $x^{m_1+m_2+\ldots+m_k}[x,a] = 0$ for sufficiently large k and $a \in Z$, by Lemma 1. Thus $N \subseteq Z$ and in view of (II-A) R satisfies (I-Z). Hence R is commutative by Herstein's theorem [8].

Again if R satisfies (V), then by using the same arguments as above we get the required result.

Remarks 1. The following example suggests that it is essential to retain any one of the conditions (IV) and (V) together with (II-A) in the hypotheses of above theorem in order to get the commutativity of R.

Example 1. Let $R = \left\{aI + B/B = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a,b,c,d \in GF(2)\right\}$. Obviously $N = \{B\}$ and R satisfies neither (IV) nor (V). If we assume that $A = N$, then for any x in R, $x - x^2f(x) \in A$. However, R is not commutative.

2. The justification for unity 1 in the hypotheses of our theorem may be given by the following example.

Example 2. Let $R = \left\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} /a,b,c \in GF(2)\right\}$. It can be easily seen that R is a nilpotent ring of index 3. Thus $R = N$ and R satisfies both the conditions (IV) & (V). Next, if we assume that $A = N$ then R also satisfies condition (II-A). However, R is not commutative.

3. Although, the above example strengthens the existence of unity 1 in the hypotheses of our theorem, nevertheless, the same result may be extended in a more general setting.

Theorem 2. *Let R be a left (resp. right) s-unital ring satisfying (IV) (resp. V). Suppose, further that there exists a subset A of N for which R satisfies (II-A). Then R is commutative (and conversely).*

The following lemma is essentially proved in [13].

Lemma 2. *Let R be a right (resp. left) s-unital ring. If for each pair of elements x, y of R there exist a positive integer $k = k(x,y)$ and an element $e' = e'(x,y)$ of R such that $e'x^k = x^k$ and $e'y^k = y^k$ (resp. $x^ke' = x^k$ and $y^ke' = y^k$), then R is s-unital.*

Proof of Theorem 2. Since R is left (resp. right) s-untial then for any x, y in R, we can find an element e of R such that $ex = x$ and $ey = y$ (resp. $xe = x$ and $ye = y$). Thus there exist integers $m = m(x,e) \geq 0$, $n = n(x,e) > 1$, $p = p(x,e) \geq 0$ and $q = q(x,e) \geq 0$ such that

$$x^{m+1}e = [x,\ x^me - x^pe^nx^q] + x^{m+1} = x^{m+1}$$

$$(\text{resp.} \quad ex^{m+1} = [x,\ ex^m - x^pe^nx^q] + ex^{m+1} = x^{m+1}).$$

Similarly, $y^{m'+1}e = y^{m'+1}$ (resp. $ey^{m'+1} = y^{m'+1}$). Hence $x^{m+m'+1}e = x^{m+m'+1}$, $y^{m+m'+1}e = y^{m+m'+1}$ (resp. $ex^{m+m'+1} = x^{m+m'+1}$, $ey^{m+m'+1} = y^{m+m'+1}$) and in view of Lemma 2, R is s-unital. Thus by [9, Proposition 1], we may assume that R has unity 1 and hence R is commutative by Theorem 1.

Remark 4. The following example shows that there are noncommutative left (resp. right) s-unital rings satisfying (V) (resp. (IV)).

Example 3. Let

$$R_1 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$$

and

$$R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be two subrings of 2×2 matrices over $GF(2)$. Obviously in both the cases N is the set consisting of the matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Further if $A = N$, then for any x in R we can find a polynomial $f(t)$, for example $f(t) = t$, such that $x - x^2 f(x) \in A$. Also R_1 (resp. R_2) is a left (resp. right) s-unital ring and for any fixed integers $m \geq 0$, $n > 1$, $p \geq 0$, $q \geq 0$, R_1 (resp. R_2) satisfies the condition (V) (resp. (IV)).

3. Property (II-A) For Commutative A

Motivated by Theorem 1 of Tominaga and Yaqub [12], we derive the following:

Theorem 3. *Let R be a normal ring, and let A be a nil commutative subset of R for which R satisfies $(II - A)$. Then R is commutative.*

In the proof we shall use the following lemma, the proof of which is contained in that of [11].

Lemma 3(i). *Let ϕ be a ring homomorphism of R onto R^*. If R satisfies $(I - A)$, $(II-A)$ or $(III-A)$, then R^* satisfies $(I-\phi(A))$, $(II-\phi(A))$ or $(III-\phi(A))$ respectively.*

(ii). *If there exists a commutative subset A of N for which R satisfies (II-A) and (III-A), then R is commutative.*

(iii). *If A is commutative and R satisfies $(II - A)$, then N is a commutative nil ideal of R containing a commutator ideal of R and contained in a centralizer of A, in particular, $N^2 \subseteq Z$.*

Proof of Theorem 3. In view of Lemma 3(i), R can be assume to be subdiractly irreducible. Let x be an arbitrary element of $R \setminus Z$. By using hypotheses (II-A), we find that there exists $y \in < x >$ and a positive integer m such that $x^m = x^{m+1}y$. Obviously, $e = x^m y^m$ is an idempotent with $x^m = x^m e$. Since idempotents of R are central, hence e is either 0 or 1. But R has no unity, hence $e = 0$ and by Lemma 3(iii) x is in the commutative ideal N and so $[[a, x], x] = 0$ for all $a \in A$. Hence R is commutative by Lemma 3(ii).

Remarks 6. Example 3 also shows that the condition (II-A) alone does not imply commutativity of rings in the above theorem.

7. In retrospect, it is tempting to conjecture as follows:

Conjecture. Let R be a ring satisfying any one of the conditions (IV) and (V). Further, if there exists a nil commutative subset A for which R satisfies (II-A), then R is commutative.

8. A careful observation of the proof of Theorem 3 shows that the above conjecture is true if R is normal. However, Example 3, violates the above conjecture because the centrality of idempotents in R_1 (resp. R_2) are not implied by the condition (V) (resp. (IV)) together with (II-A).

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ROTARU STARLIKE INTEGRAL OPERATORS

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Abstract. Let $S^*(a,b)$ denote the class of analytic functions f in the unit disc U, with $f(0)=f'(0)-1=0$, satisfying the condition $|(zf'(z)/f(z))-a|<b$, $z\in U$, where $a\in C$, $|a-1|<b\le Re(a)$. In this paper we obtain integral operators which map $S^*(a,b)$ into $S^*(a,b)$ and $S^*(\lambda)\times S^*(a,b)$ into $S^*(\lambda)$.

1. Introduction.

Let V denote the class of functions f analytic in the unit disc U, with $f(0)=f'(0)-1=0$. A function f of V is said to belong to $S^*(\lambda)$, the class of starlike functions of order λ, if $Re(zf'(z)/f(z))>\lambda$, for $z\in U$, $0\le\lambda<1$. The class S^* of starlike functions is identified by $S^*(0)\equiv S^*$. In [2], Rotaru investigated properties of the class $S^*(a,b)$ of functions $f\in V$ satisfying $|(zf'(z)/f(z))-a|<b$, $z\in U$, where $a\in C$, $|a-1|<b\le Re(a)$. It is clear that $S^*(a,b)\subset S^*(Re(a)-b)\subset S^*$.

Recently in [3], Vinod kumar and Shukla have studied the integral operators of the form

$$I(f)=[\frac{\gamma+\beta}{z^\gamma}\int_0^z t^{\delta-1}f^\alpha(t)\,dt]^{1/\beta} \tag{1.1}$$

and

$$I(f,g)=[\frac{\gamma+\beta+\sigma}{z^{\gamma+\sigma}}\int_0^z t^{\delta-1}f^\alpha(t)g^\sigma(t)\,dt]^{1/\beta}, \tag{1.2}$$

where $\alpha,\beta,\gamma,\delta$ and σ are real constants and f and g belong to some favoured classes of univalent functions. By imposing suitable restrictions on $\alpha,\beta,\gamma,\delta$ and σ thay have shown that, for $a=\overline{a}$, $I(f)$ maps $S^*(a,b)$ into itself and, for $a=\overline{a}$, $I(f,g)$ maps $S^*(\lambda)\times S^*(a,b)$ into $S^*(\lambda)$.

In the present paper we prove that, for $a\in C$, $I(f)$ maps $S^*(a,b)$ into itself and also, for $a\in C$, $I(f,g)$ maps $S^*(\lambda)\times S^*(a,b)$ into $S^*(\lambda)$.

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2. Preliminary Lemmas

The following lemma may be easily proved by using Schwarz's lemma as in Rotaru [2].

Lemma 2.1. *The function f belongs to $S^*(a,b)$ if and only if there exists a function w analytic in U satisfying $w(0)=0$, $|w(z)|<1$ for $z \in U$ such that*

$$\frac{zf'(z)}{f(z)} = \frac{1+A\,w(z)}{1+\overline{B}\,w(z)}, \qquad z \in U,$$

where $A=(b^2-|a|^2+a)/b$ and $B=(1-a)/b$.

Next we have the well known Jack's lemma [1].

Lemma 2.2. *Let w be analytic in U with $w(0)=0$. If there exists a $z_0 \in U$ such that $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|$, then $z_0 w'(z_0)=k\,w(z_0)$ for some $k \geq 1$.*

Lastly we prove a lemma which plays an important role in establishing our main result.

Lemma 2.3. *Let α,β,b be real numbers and 'a' be a complex number such that $0<\alpha\leq\beta$, $|a-1|<b\leq Re(a)$. If $d=(a\alpha+\beta-\alpha)/\beta$ and $e=b\alpha/\beta$, then $S^*(d,e)\subset S^*(a,b)$.*

Proof. We need only to consider the case $\alpha<\beta$. In order to establish the lemma it suffices to show that

$$Re(a)-b < Re(d)-e \quad \text{and} \quad Re(d)+e < Re(a)+b. \tag{2.1}$$

Let $Re(a)-b\geq Re(d)-e$. Then $Re(a)-b\geq 1$, which is contrary to $|a-1|<b$. Next, suppose that $Re(d)+e\geq Re(a)+b$. Then $1\geq Re(a)+b$, which is also contrary to $|a-1|<b$. Therefore inequalities in (2.1) hold and hence the required result follows.

From now on d and e will be as in lemma 2.3.

3. Integral Operators That Map $S^*(a,b)$ into $S^*(a,b)$

An integral operator which is defined on $S^*(a,b)$ and maps $S^*(a,b)$ into (onto) itself is called Rotaru starlike integral operator.

We now prove the following:

Theorem 3.1. *Let α,β,γ and δ be real constants such that $0<\alpha\leq\beta$ and $\gamma+\beta=\delta+\alpha$. If $f(z)\in S^*(a,b)$ then the function $F(z)$ defined by*

$$F(z) = \Big[\frac{\gamma+\beta}{z^\gamma}\int_0^z t^{\delta-1}f^\alpha(t)\,dt\Big]^{1/\beta} \tag{3.1}$$

also belongs to $S^*(a,b)$, *provided* $\frac{c}{4|Im(a)|} > \gamma+\beta \geq \alpha c\frac{Re(1-B)}{|1-B|^2}$, *where* $c = (b^2-|a-1|^2)/b$ *and* $B = (1-a)/b$.

Powers in (3.1) are meant as principal values.

Proof. First we prove that $F(z) \in S^*(d,e)$. Let $w(z)$ be a regular function defined in U by

$$\frac{zF'(z)}{F(z)} = \frac{1+D\,w(z)}{1+\overline{E}\,w(z)} \tag{3.2}$$

where $D = (e^2-|d|^2+d)/e$, $E = (1-d)/e$. Clearly $w(0) = 0$ and $w(z) \neq -1/\overline{E}$. From (3.1) and (3.2) we obtain

$$Nz^{\delta-\gamma}\frac{f^\alpha(z)}{F^\beta(z)} = \frac{N+M\,w(z)}{1+\overline{E}\,w(z)}, \tag{3.3}$$

where $N = \gamma+\beta$, $M = D\beta+\overline{E}\gamma$. Logarithmic differentiation of (3.3) yields

$$\frac{\alpha}{\beta}[\frac{zf'(z)}{f(z)} - a] = \frac{Ee+e\,w(z)}{1+\overline{E}\,w(z)} + \frac{m\,zw'(z)}{(1+\overline{E}\,w(z))(N+M\,w(z))} \tag{3.4}$$

where $m = (e^2-|d-1|^2)/e = \frac{\alpha}{\beta}(b^2-|a-1|^2)/b = \frac{\alpha}{\beta}\,c > 0$.

Now we claim that $|w(z)| < 1$, for otherwise by lemma 2.2 there exists a $z_0 \in U$ such that $|w(z_0)| = 1$ and

$$z_0w'(z_0) = kw(z_0), \qquad k \geq 1. \tag{3.5}$$

From (3.4) and (3.5) we have

$$\frac{\alpha}{\beta}[\frac{z_0f'(z_0)}{f(z_0)} - a] = \frac{P(z_0)}{Q(z_0)} \tag{3.6}$$

where

$$P(z_0) = EeN + (EeM+eN+km)\,w(z_0) + eM\,w^2(z_0)$$

and

$$Q(z_0) = N + (\overline{E}N+M)\,w(z_0) + \overline{E}M\,w^2(z_0).$$

Clearly

$$|\,P(z_0)\,|^2 - e^2\,|\,Q(z_0)\,|^2 > 0 \qquad \text{provided} \tag{3.7}$$
$$\frac{km}{2e} + N + Re(EM) + Re((\overline{E}N+M)\,w(z_0)) > 0$$

or equivalently

$$\frac{(k-1)m}{2e} + A_0 + B_0\,Re(w(z_0)) - D_0\,Im(w(z_0)) + \frac{m}{2e} > 0, \tag{3.8}$$

where $A_0 = Re(N + EM)$, $B_0 = Re(\overline{E}N + M)$ and $D_0 = Im(\overline{E}N + M)$. Since $k \geq 1$ and $m > 0$, (3.8) holds if $A_0 \pm B_0 \geq 0$ and $\frac{m}{2e} \pm D_0 > 0$.
Now

$$\begin{aligned} A_0 \pm B_0 &= Re[\{(\gamma+\beta) + E(D\beta + \overline{E}\gamma)\} \pm \{\overline{E}(\gamma+\beta) + (D\beta + \overline{E}\gamma)\}] \\ &= Re[(1 \pm \overline{E})(\gamma+\beta) \pm (1 \pm E)(D\beta + \overline{E}\gamma)] \\ &= Re[\mid 1 \pm E \mid^2 \gamma + \beta(1 \pm E)(1 \pm D)] \\ &\geq 0, \text{ provided } \gamma \geq -\frac{\beta Re(1 \pm E)(1 \pm D)}{\mid 1 \pm E \mid^2} \\ &\qquad = -(\beta \pm \alpha c \frac{Re(1 \pm B)}{\mid 1 \pm B \mid^2}), \text{ where } B = (1-a)/b. \end{aligned}$$

and

$$\frac{m}{2e} \pm D_0 = \frac{m}{2e} \pm 2\frac{(\gamma+\beta)}{e} Im(d) > 0 \text{ provided } \gamma + \beta < \frac{m}{4 \mid Im(d) \mid} = \frac{c}{4 \mid Im(a) \mid}.$$

Thus from (3.6) and (3.7) it follows in view of $e = \alpha b/\beta$ that $|(z_0 f'(z_0)/f(z_0)) - a| > b$ provided

$$\frac{c}{4 \mid Im(a) \mid} > \gamma + \beta \geq \alpha c \frac{Re(1-B)}{\mid 1 - B \mid^2}.$$

But this is contrary to the fact that $f \in S^*(a, b)$. Therefore $|w(z)| < 1$ for z in U. Thus from (3.2) and lemma 2.1, $F \in S^*(d, e)$. Hence from lemma 2.3, $F \in S^*(a, b)$.

Corollary 3.1. *If $\frac{c}{4|Im(a)|} > 1 \geq \alpha c \frac{Re(1-B)}{|1-B|^2}$, where $c = (b^2 - |a-1|^2)/b$, $B = (1-a)/b$ and if $f \in S^*(a, b)$, then the function F defined by*

$$F(z) = \left[z^{\beta-1} \int_0^z (\frac{f(t)}{t})^\alpha \, dt\right]^{1/\beta}$$

also belongs to $S^*(a, b)$.

The above corollary follows by taking $\gamma = 1 - \beta$ and $\delta = 1 - \alpha$ in theorem 3.1.

Corollary 3.2. *Let α and η be real constants such that $\alpha > 0$, $\eta \geq 0$. If $f \in S^*(a, b)$, then the function F defined by*

$$F(z) = \left[\frac{\gamma + \alpha + \eta}{z^\gamma} \int_0^z t^{\gamma+\eta-1} f^\alpha(t) \, dt\right]^{1/(\alpha+\eta)}$$

also belongs to $S^*(a, b)$, provided $\frac{c}{4|Im(a)|} > \alpha + \eta + \gamma \geq \alpha c \frac{Re(1-B)}{|1-B|^2}$, where $c = (b^2 - |a - 1|^2)/b$ and $B = (1-a)/b$.

The above result is obtained by setting $\beta = \alpha + \eta$ and $\delta = \gamma + \eta$ in theorem 3.1.

Remark. For $a = \overline{a}$, theorem 3.1, corollary3.1 and corollary 3.2 of Vinod kumar and Shukla [3] are obtained as particular cases from our results.

We now consider the integral operator defined in (3.1) in a limiting case. When $\alpha = \beta$, the relation (3.1) can be written as

$$f(z) = F(z)([\gamma + (\beta zF'(z))/F(z)]/(\gamma+\beta))^{1/\beta}.$$

when $\beta \to 0$, the above relation reduces to

$$f(z) = F(z)\exp[\{(zF'(z)/F(z)) - 1\}/\gamma] \tag{3.9}$$

where $\gamma > 0$. It follows from (3.9) that

$$F(z) = f(z)\exp\Big[-z^{-\gamma}\int_0^z t^{\gamma-1}\{t\frac{f'(t)}{f(t)} - 1\}\,dt\Big]. \tag{3.10}$$

We now take γ in (3.10) a complex number with $Re(\gamma) > 0$ and prove the following.

Theorem 3.2. *If $f(z) \in S^*(a,b)$ and $\gamma \in C$ such that $Re(\gamma) > 0$, then the function F defined by* (3.10) *also belongs to $S^*(a,b)$.*

Proof. Let $w(z)$ be a regular function defined in U by

$$\frac{zF'(z)}{F(z)} = \frac{1 + A\,w(z)}{1 + \overline{B}\,w(z)}, \tag{3.11}$$

where $A = (b^2 - |a|^2 + a)/b$ and $B = (1-a)/b$. Evidently $w(0) = 0$ and $w(z) \neq -1/\overline{B}$ for z in U. Differentiating (3.9) and using (3.11) we get

$$\frac{zf'(z)}{f(z)} - a = \frac{bB + b\,w(z)}{1 + \overline{B}\,w(z)} + \frac{(c/\gamma)\,zw'(z)}{(1+\overline{B}\,w(z))^2}, \tag{3.12}$$

where $c = (b^2 - |a-1|^2)/b > 0$. We shall prove that $|w(z)| < 1$, $z \in U$. For, if not, there exists a $z_0 \in U$, by lemma 2.2, such that $|w(z_0)| = 1$ and

$$z_0 w'(z_0) = kw(z_0), \qquad k \geq 1. \tag{3.13}$$

From (3.12) and (3.13) we obtain

$$\frac{z_0 f'(z_0)}{f(z_0)} - a = b[\frac{B + w(z_0) + \Phi(z_0)w(z_0)}{1 + \overline{B}w(z_0)}]$$

where $\Phi(z_0) = \dfrac{kc}{b\gamma(1+\overline{B}w(z_0)}$,

Now $|\dfrac{z_0 f'(z_0)}{f(z_0)} - a| > b$ provided $|\, B + w(z_0) + \Phi(z_0)w(z_0)\,|^2 > |\, 1 + \overline{B}w(z_0)\,|^2$. This condition reduces to the following:

$$|\,\Phi(z_0)\,|^2 + 2Re[(1 + \overline{B}w(z_0)\Phi(z_0)] > 0,$$

which is true since $Re[1+\overline{B}w(z_0)\Phi(z_0)] = Re(\frac{kc}{b\gamma}) > 0$. But this is a contradiction to the hypothesis that $f \in S^*(a,b)$. Hence $|w(z)| < 1$ for $z \in U$ and from (3.11) we conclude that $F \in S^*(a,b)$.

Remark. For $a = \overline{a}$, the above theorem improves a recent result of Vinodkumar and Shukla [3] who proved it when γ is a real number. Here it is worth noting that the technique used by them with the help of Jack's lemma, fails when γ is a complex number.

4. Integral Operators that Map $S^*(\lambda) \times S^*(a,b)$ into $S^*(\lambda)$

Theorem 4.1. *Let $\alpha, \beta, \gamma, \delta$ and σ be real constants such that $\alpha > 0$, $\beta \geq \alpha$, $\sigma \geq 0$, $\alpha + \delta = \beta + \gamma$ and $\gamma + \sigma + \lambda\beta > 0$. If $f \in S^*(\lambda)$ and $g \in S^*(a,b)$, $(a,b) \in R = \{(a,b) : |a-1| < b \leq a^*\}$, where $a^* = min\{Re(a), (Re(a)-1) + \frac{\beta(1-\lambda)}{2\sigma(\gamma+\sigma+\lambda\beta)}\}$, then the function F defined by*

$$F(z) = [\frac{\gamma+\beta+\sigma}{z^{\gamma+\sigma}} \int_0^z t^{\delta-1} f^\alpha(t) g^\sigma(t)\, dt]^{1/\beta} \tag{4.1}$$

also belongs to $S^*(\lambda)$.

Powers in (4.1) are meant as principal values.

Proof. Define a regular function $w(z)$ in U by

$$\frac{zF'(z)}{F(z)} = \frac{1+(2\lambda-1)\,w(z)}{1+w(z)}. \tag{4.2}$$

Clearly $w(0) = 0$ and $w(z) \neq -1$ in U. From (4.1) and (4.2) we have

$$N_1 z^{\delta-\gamma-\sigma}\{\frac{f^\alpha(z) g^\sigma(z)}{F^\beta(z)}\} = \frac{N_1 + M_1\, w(z)}{1+w(z)}, \tag{4.3}$$

where $N_1 = \gamma + \beta + \sigma$ and $M_1 = \gamma + \sigma - \beta + 2\lambda\beta$.

Logarithmic differentiation of (4.3) yields

$$\begin{aligned} \frac{zf'(z)}{f(z)} &= \frac{\sigma}{\alpha}(1-a) - \frac{\sigma}{\alpha}\{\frac{zg'(z)}{g(z)} - a\} + \frac{\beta}{\alpha}\{\frac{1+(2\lambda-1)w(z)}{1+w(z)}\} \\ &\quad - \{\frac{\beta-\alpha}{\alpha}\} - \frac{2\beta(1-\lambda)zw'(z)}{\alpha(1+w(z))(N_1+M_1 w(z))}. \end{aligned} \tag{4.4}$$

Now we claim that $|w(z)| < 1$, for otherwise by lemma 2.2 there exists a $z_0 \in U$ such that $|w(z_0)| = 1$ and

$$z_0 w'(z_0) = kw(z_0), \qquad k \geq 1. \tag{4.5}$$

Using the technique similar to the one employed in the proof of theorem 4.1 of Vinod kumar and Shukla [3], we obtain from (4.4) and (4.5)

$$\begin{aligned} Re\{z_0\frac{f'(z_0)}{f(z_0)}\} &\leq \lambda + \frac{2(\gamma+\sigma+\lambda\beta)\{2\sigma(\gamma+\sigma+\lambda\beta)[b-(Re(a)-1)] - \beta(1-\lambda)\}}{\alpha[N_1^2 + 2N_1M_1Re(w(z_0)) + M_1^2]} \\ &\leq \lambda, \text{ provided } b \leq (Re(a)-1) + \frac{\beta(1-\lambda)}{2\sigma(\gamma+\sigma+\lambda\beta)}. \end{aligned}$$

But this is contrary to the fact that $f \in S^*(\lambda)$. Therefore $|w(z)| < 1$ and hence, from (4.2), $F \in S^*(\lambda)$.

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