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SUPER QUASI-ADEQUATE SEMIGROUPS

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Let S be a semigroup and let L denote Green's relation on S, For $a, b \in S$, let $(a, b) \in L^*$ if and only if $(a, b) \in L$ in some cversemigroup of S. R^* is defined dually and let $H^* = L^* \cap R^*$. From ([11] or [12]), $(a, b) \in L^*$ if and only if, for all x, $y \in S^1$ (S with an appended identity), $ax = ay$ if and only if $bx = by$. So L^* is a right congruence relation and R^* is a left congruence relation. Fountain [9] terms a semigroup S *abundant* if each L^*-class of S and each R^*-class of S contains an idempotent, and Fountain [9] terms S *superabundant* if each H^*-class of S contains an idempotent. If S is a regular semigroup, $L^* = L$ and $R^* = R$. Hence, regular semigroups are abundant semigroups and unions of groups are superabundant semigroups.

In [9], Fountain gave superabundant analogues to the Rees Theorem and Clifford's well known theorem that a semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups. In [7], El-Qallali terms an abundant semigroup S to be L^*-unipotent if $E(S)$, the set of idempotents of S, form a subsemigroup and each L^*-class of S contains precisely one idempotent. In [7], El-Qallali gives a structure theorem for super L^*-unipotent semigroups on which H^* is a congruence (L^*-unipotent bands of cancellative monoids [7]). A semigroup S is termed L-unipotent if each L-class of S contains precisely one idempotent (equivalently, S is orthodox and each J-class of $E(S)$ is a right zero semigroup [20]). El-Qallali's theorem is a superabundant analogue to Bailes' structure theorem for L-unipotent union of groups on which H is a congruence (L-unipotent bands of groups) [1].

Let S be an abundant semigroup. Fountain [8] terms S an *adequate semigroup* if $E(S)$ is a semilattice. El-Qallali and Fountain [6] terms S a quasi-adequate semigroup if $E(S)$ is a subsemigroup. If, furthermore, L is a congruence relation on $E(S)$, we term S a generalized L^*-unipotent semigroup. El-Qallali and Fountain [5], term a congruence e on S *good* if $a\,L^*b$ implies aeL^*be and aR^*b implies aeR^*be.

In section 1, we give a structure theorem for super quasi adequate semigroups S (Theorem 1.11). We first specialize the above mentioned results of Fountain to super quasi-adequate semigroups S. In particular, S is a semilattice Y of semigroups (S_y : $y \in Y$) where $S_y = T_y \times E(S_y)$ (algebraic direct product) where T_y is a cancellative monoid and $E(S_y)$ is a rectangular band (Lemma 1.1). For $(g; i, j)$, $(h; r, s) \in S$, define

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$(g;i,j)\delta(h;r,s)$ if $(g;i,j)$, $(h;r,s) \in S_y$, say, and $g = h$. Then, δ is the minimum adequate good congruence on S (Proposition 1.3) and S/δ is a strong semilattice Y of the T_y (Lemma 1.4). Then, S^1 divides $V \circ (\widehat{S/\delta})^1$ where V is an L-trivial and idempotent monoid, $\circ$ is wreath product, $\widehat{\ }$ is the Rhodes expansion, $(\widehat{S/\delta})$ is a semilattice Y of left cancellative semigroups $(X_y : y \in Y)$ with idempotents, and $E((\widehat{S/\delta}))$ is a semilattice Y of right zero semigroups $(E(X_y) : y \in Y)$ (Theorem 1.11). If S is an orthodox union of groups, δ becomes the smallest inverse semigroup congruence on S, T_y becomes a maximal subgroup of S, and $X_y = T_y \times E(X_y)$ (algebraic direct product) (see Lemma 1.12). Hence, Theorem 1.11 is a superabundant semigroup analogue to our structure theorem for orthodox unions of groups [26].

In section 2, we give a structure theorem for super generalized L^*-unipotent semigroups S (Theorem 2.4). We first show that $\delta \cap L$ is the smallest L^*-unipotent good congruence on S and $S/\delta \cap L$ is a semilattice Y of the semigroups $((T_y \times J_y) : y \in Y)$ were J_y is an R-class of $E(S_y)$ (Proposition 2.1). Then,

$$S \leq W^1 \circ (E(S)/L)^1 \circ (S/\delta \cap L)^1$$
$$(\leq \text{ means "is embedded in"}) \text{ and } S/\delta \cap L$$
$$\leq (S/\delta \cap L/e)^1 \overset{\Omega}{\circ} (E(S)/L)^1$$

where W is a lower partial chain Y of left zero subsemigroups of $E(S)$, e is the smallest adequate good congruence on $S/\delta \cap L$, $S/\delta \cap L/e$ is a strong semilattice Y of the T_y, and $\overset{\Omega}{\circ}$ is reverse wreath product (Theorem 2.4). An orthodox semigroup S is termed generalized L-unipotent if L is a congruence relation on $E(S)$. If S is a generalized L-unipotent union of groups, $\delta \cap L$ becomes the smallest L-unipotent congruence on S. Hence, Theorem 2.4 is a superabundant analogue to our structure theorem for generalized L-unipotent unions of groups [24].

In section 3, we show that if S is a super R^*-unipotent semigroup, then $S \leq (E(S))^1 \circ (S/\delta)^1$ where $E(S)$ is a semilattice Y of left zero semigroups (Theorem 3.1). Theorem 3.1 is a superabundant analogue to our structure theorem for R-unipotent unions of groups [24].

Abundant semigroup analogues to many theorems in regular semigroup theory have been given by Fountain ([8], [9]), El-Qallali and Fountain ([5], [6]), and El-Qallali [7].

We have studied the structure of generalized L-unipotent semigroups in ([21], [22], [23], [24]), R-unipotent semigroups have been studied extensively by many authors-most recently by Szendrei ([14], [15]).

A submonoid of a monoid S is a subsemigroup of S containing the identity of S.

A semigroup (monoid) S is said to *divide a semigroup (monoid)* T if there exists a homomorphism of a subsemigroup (submonoid) of T onto S. We also say T covers S in this case and write $S < T$. If there exists an isomorphism of S into T, we write $S \leq T$. R, L, H, D and J will denote Green's relations and $E(S)$ will denote the set of idempotents of a semigroup S.

See [9] for the definition of J^*. If S is a regular semigroup $J^* = J$.

We adopt the following notation and definitions from [24, p. 181-182]: S^1 (S with appended identity), $S^\bullet$, wreath product "$\circ$" of semigroups, reverse wreath product "$\overset{\Omega}{\circ}$" of semigroups, type A semigroup congruence (for example, inverse semigroup congruence), ae ($a \in S$, a semigroup) (e, a congruence on S, will also denote the natural homomorphism of S onto S/e), and unions of groups.

For other definitions not given in this paper, see [2] or [10]. We also adopt the notation of [2] unless otherwise specified.

A monoid S is termed L-trivial and idempotent if each L-class of S is a singleton and S is a band.

Section 1 - The Structure of Super Quasi-Adequate Semigroups

In this section, we describe the minimum adequate good congruence δ on a super quasi-adequate semigroup (Proposition 1.3 and Lemma 1.4) and give a structure theorem for super quasi-adequate semigroups (Theorem 1.11).

Let S be a semigroup. For ${}_a \in S$, L^*_a or $L^*_a(S)$ (in case of ambiguity) will denote the L^*-class of S containing ${}_a$ (notation of [9]).

Let S be a semigroup and I and J be sets and let $P : J \times I \longrightarrow S$ with $({}_{j,i})\ P = p_{ji}$. Let $M(S, I, J, P)$ denote $S \times I \times J$ under the multiplication $(a;_{i,j})(b;_{r,s}) = (a\, p_{jr}\, b;_{i,s})$. We term $M(S, I, J, P)$ a Rees Matrix semigroup over S with entries in P.

The following lemma gives the "gross" structure of super quasi-adequate semigroups.

Lemma 1.1. *A semigroup S is super quasi-adequate if and only if S is a semilattice $Y = S/J^*$ of semigroups $(S_y :_y\in Y)$ where $S_y = T_y \times E(S_y)$ where T_y is a cancellative monoid and $E(S_y)$ is a rectangular band, $L^*_a(S) = L^*_a(S_y)$ and $R^*_a(S) = R^*_a(S_y)$ for ${}_y \in Y$ and ${}_a \in S_y$ and $E(S)$ is a semilattice Y of rectangular bands $(E(S_y) :_y\in Y)$.*

Proof. Utilizing [9, Theorem 6.8 and its proof and Corollary 5.2], we obtain the above theorem (except the statement about $E(S)$) with $S_y = M(T_y, I_y, J_y, P_y)$, a Rees matrix semigroup over a cancellative monoid T_y where the entries of P_y are units U of T_y. As is easily shown, [2, Lemma 3.6] is valid for the above matrix semigroups if we require the mappings to have range U. Using this Lemma, we may "normalize" P_y such that all the elements in a given row and a given column are the identity e of T_y. Then, using the assumption that $E(S)$ is a subsemigroup, we may show $p_{ji} = e$ for all $j \in J_y$ and $i \in I_y$. Hence, $M(T_y, I_y, J_y, P_y) = T_y \times E(S_y)$ where $E(S_y)$ is a rectangular band.

To show δ is a congruence relation (Proposition 1.3), we will need the following lemma.

Lemma 1.2. *Let $S_y = T_y \times E_y$ and $S_x = T_x \times I_x \times J_x$ where T_y and T_x are cancellative monoids, E_y is a rectangular band, I_x is a left zero semigroup, and J_x is a*

right zero semigroup. Assume these exists
(a) a left representation $_A \longrightarrow \lambda_A$ *of* S_y *by transformations of* I_x
(b) a right representation $_A \longrightarrow e_A$ *of* S_y *by transformations of* J_x
(c) a homomorphism ϕ *of* T_y *into* T_x,

Define a binary operation on $S_y \cup S_x$ *extending the given ones on* S_y *and* S_x *by defining products of* $_A = (a, e) \in S_y$ *and* $(b; i, j) \in S_x$ *as follows:*

$$\begin{aligned}(a,e)(b;i,j) &= (a\phi b; \lambda_A i, j)\\ (b;i,j)(a,e) &= (b(a\phi); i, je_A).\end{aligned}$$

Then, $S_y \cup S_x$ *becomes a semigroup with* S_x *an ideal.*

Conversely every possible binary associative operation on $S_y \cup S_x$ *extending the given ones on* S_y *and* S_x*, and such that* S_x *is an ideal, can be constructed in the above manner.*

Proof. Lemma 1.2 has been established by Clifford [3, Lemma 2.5] in the case T_y and T_x are groups. Clifford's proof is easily seen to be valid when T_y and T_x are just cancellative monoids.

Proposition 1.3. *Let* S *be a super quasi-adequate semigroup. Then,* δ *is the minimum adequate good congruence on* S.

Proof. We first show that δ is a congruence relation on S. Let $\overline{\delta}$ denote the smallest congruence on S containing δ. Suppose a $\overline{\delta}\, b$. Then, there exists $a = a_1, a_2, \ldots, a_n = b \in S$ such that $a_i = x_i u_i y_i$, $a_{i+1} = x_i v_i y_i$ where $x_i, y_i \in S^1$ and $(u_i, v_i) \in \delta$ for $1 \le i \le n-1$. Let $x_i = (w; i, j)_\alpha \in S_\alpha$, $y_i = (h; r, s)_\beta \in S_\beta$, $u_i = (g; m, n)_\gamma$, and $v_i = (g; c, d)_\gamma$. Hence, $a_i = (A; p, q)_{\alpha\beta\gamma} \in S_{\alpha\beta\gamma}$ and $a_{i+1} = (B; k, l)_{\alpha\beta\gamma} \in S_{\alpha\beta\gamma}$ say. Let $\theta =_{\alpha\beta\gamma}$. Thus,

$$\begin{aligned}(A;p,q)_\theta &= (w;i,j)_\alpha (g;m,n)_\gamma (h;r,s)_\beta\\ (B;k,l)_\theta &= (w;i,j)_\alpha (g;c,d)_\gamma (h;r,s)_\beta\end{aligned}$$

Multiply both of the above equations on the left and right by $(e; p, q)_\theta$ where e is the identity of T_θ.
Hence,

$$\begin{aligned}(A;p,q)_\theta &= (\overline{W};\overline{i},\overline{j})_\theta (g;m,n)_\gamma (\overline{h};\overline{r},\overline{s})_\theta\\ (B;p,q)_\theta &= (\overline{W};\overline{i},\overline{j})_\theta (g;c,d)_\gamma (\overline{h};\overline{r},\overline{s})_\theta\end{aligned}$$

say,
Using Lemma 1.2

$$(A;p,q)_\theta \;=\; (\overline{W}(g\omega_{\gamma,\theta}); \overline{i}, \overline{j}e_{(g,m,n)_\gamma})_\theta (\overline{h}; r, s)_\theta = \;(\overline{W}(g\omega_{\gamma,\theta})\overline{h}; \overline{i}, s)_\theta.$$

where $\omega_{\gamma,\theta}$ is the homomorphism of T_γ into T_θ given by Lemma 1.2 and $(B; p, q)_\theta = (\overline{W}(g\omega_{\gamma,\theta}); \overline{i}, \overline{j}e_{(g,c,d)_\gamma})_\theta (\overline{h}; r, s)_\theta = (\overline{W}(g\omega_{\gamma,\theta})\overline{h}; \overline{i}, s)_\theta$. Hence, $A = B$. Thus $a_i \delta a_{i+1}$ for $1 \le i \le n-1$. Hence, $a\delta b$. Thus, $\overline{\delta} = \delta$, and, hence, δ is a congruence on S.

Let $a \in S$ and let $a^+, a^* \in E(S)$ such that $a^+ R^* a$ and $a^* L^* a$. Using [9, Corollary 6.2 and Proposition 6.5] and Lemma 1.1, $a^+, a^*, a \in S_y$, say. Hence, using [6, Corollary 2.4 and Proposition 2.6], δ is the minimum adequate good congruence on S.

Lemma 1.4. *Let S be a super quasi-adequate semigroup. Then, S/δ is a strong semilattice Y of cancellative monoids $(T_y;_y \in Y)$.*

Proof. Let $(\overline{g;i,j})$ denote the δ-class of S containing $(g;i,j)$. Since $(\overline{g;i,j})\tau = g$ defines a 1-1 map of S/δ onto $T = U(T_y :_y \in Y)$, T becomes a groupoid under the multiplication $ab = (a\tau^{-1}b\tau^{-1})\tau$ and τ defines an isomorphism of S/δ onto T. If $g, h \in T_y$, $gh = ((\overline{g;i,j})(\overline{h;k,e}))\tau = (\overline{gh;i,e})\tau = gh$ (the last product is multiplication in T_y). Hence, T is a semilattice Y of cancellative monoids $(T_y :_y \in Y)$. For $a \in T_x$ and $_x \geq _y$, define a $\varsigma_{x,y} = ae_y$ where e_y is the identity of T_y. It is routine to verify that $\varsigma_{x,y}$ is a homomorphism of T_x into T_y, $\varsigma_{y,y}$ is the identity map on T_y, and, for $a \in T_y$, $b \in T_x$, $ab = a\varsigma_{y,yx}b\varsigma_{x,yx}$. Using the fact that the idempotents of T commute by Proposition 1.3, its easily seen that $\varsigma_{y,x}\varsigma_{x,w} = \varsigma_{y,w}$ for $_y \geq _x \geq _w$. Hence, T is a strong semilattice $\varsigma(Y;T_y;\varsigma_{y,x})$ of cancellative monoids (notation of [10]). We identify S/δ and T.

We next describe the Rhodes expansion $\widehat{S}$ of an arbitrary semigroup S (see [17] and [13]). The Rhodes expansion and certain of its properties will be crucial in developing our structure theory of super quasi-adequate semigroups. If $a, b \in S$, $a \leq b$ means $a \cup Sa \leq b \cup Sb$ and $a < b$ means $a \leq b$ but $a \,\mathrm{L}\, b$. Let $S_+ = \{(s_n, \ldots, s_1) : s_i \in S$ for $1 \leq i \leq n$ and $s_1 \leq s_2 \leq \ldots \leq s_n\}$. If $x = (s_n, \ldots, s_1)$, $y = (t_m, \ldots, t_1)$ define $xy = (s_n t_m, \ldots, s_1 t_m, t_m, \ldots, t_1)$. Then, S_+ is a semigroup under this multiplication. If $a = (s_n, \ldots, s_1) \in S_+$ and $s_{k+1} L s_k$ for some $1 \leq k \leq n-1$ delete s_k to obtain $a_1 \in S_+$ and denote the deletion by $a \longrightarrow a_1$. Perform $a \longrightarrow a_1 \longrightarrow \ldots \longrightarrow a_k$ where $a_k = (s_n, s_{n1}, \ldots, s_{nr})$ with $s_n < s_{n1} < \ldots < s_{nr}$ (such an a_k is termed an irreducible element of S_+). Write $a_k = \text{red } a$ and $a \sim b$ if red a = red b. The equivalence relation $\sim$ is a congruence relation on S_+. Let $\widehat{S} = S_+/_\sim$. $\widehat{S}$ is termed the Rhodes expansion of S after its inventor John Rhodes. $\widehat{S}$ will be treated as the set of irreducible elements of S_+ under the multiplication $ab = \text{red } (ab)$.

Lemma 1.5. *Let S be a super quasi-adequate semigroup. Then, $\widehat{S}$ is a semilattice Y of subsemigroups $(F_y :_y \in Y)$ where $F_y = \{(a_n, a_{n-1}, \ldots, a_1) : a_n \in S_y,\ a_j \in S\}$ and $E(\widehat{S})$ is the semilattice Y of rectangular bands*

$$E(F_y) = \{((e_y;i,j), a_{n-1}, \ldots, a_1) : (e_y;i,j) \in E(S_y),\ a_j \in S\}.$$

$U = (\widehat{S/\delta})$ is a semilattice Y of left cancellative semigroups with idempotent $(X_y :_y \in Y)$ where $X_y = \{(a_n, a_{n-1}, \ldots, a_1) : a_n \in T_y,\ a_j \in S/\delta\}$. $E(U)$ is a semilattice Y of right zero semigroups $(E(X_y) :_y \in Y)$ where $E(X_y) = \{(e_y, a_{n-1}, \ldots, a_1) : e_y$, the identity of T_y, $a_j \in S/\delta\}$. For $(a_n, a_{n-1}, \ldots, a_1) \in \widehat{S}$, let $(a_n, a_{n-1}, \ldots, a_1)\widehat{\delta} = \text{red}(a_n\delta, a_{n-1}\delta, \ldots, a_1\delta)$. Then, $\widehat{\delta}$ defines a homomorphism of $\widehat{S}$ onto $(\widehat{S/\delta})$.

Proof. To establish the second sentence of the lemma, utilize Lemma 1.1 and [16, Lemma 6.7] (see also [17, Lemma 11.4] and [24, Theorem 3.1(f)]). Utilizing Lemma 1.4 and [24, Theorem 3.1(f)], it is easily checked that U is a semilattice Y of the semigroups $(X_y :_y\in Y)$ and that the fourth sentence of the lemma is valid. We next show X_y is left cancellative for $_y \in Y$. Let $(x_r, x_{r-1}, \dots, x_1)$, $(a_n, a_{n-1}, \dots, a_1)$, and $(b_s, b_{s-1}, \dots, b_1)$ be elements of X_y and suppose that $(x_r, x_{r-1}, \dots, x_1) \cdot (a_n, a_{n-1}, \dots, a_1) = (x_r, x_{r-1}, \dots, x_1) \cdot (b_s, b_{s-1}, \dots, b_1)$. Hence, $\text{red}(x_r a_n, x_{r-1}a_n, \dots, x_1 a_n, a_n, a_{n-1}, \dots, a_1) = \text{red}(x_r b_s, x_{r-1}b_s, \dots, x_1 b_s, b_s, b_{s-1}, \dots, b_1)$. Thus, $x_r a_n = x_r b_s$. Hence, since T_y is a cancellative semigroup, $a_n = b_s$. Thus, $n = s$ and $a_i = b_i$ for $1 \leq i \leq n$. The last sentence of the lemma is a consequence of [16, Proposition 6.6] (see also [17] and [24, Theorem 3.11(b)]).

In the remainder of this section, S will denote a super quasi-adequate semigroup.

If A is a semigroup and $a = (a_n, \dots, a_1) \in \widehat{A}$, let $|a| = n$. We term $|a|$ the length of a.

Lemma 1.6. *If* $z \in \widehat{S}$, $|z| = |z\widehat{\delta}|$

Proof. Let $z = (a_n, a_{n-1}, \dots, a_1)$. Suppose $a_{k+1}\delta L a_k\delta$ for some $1 \leq k \leq n-1$. Using Lemma 1.1, let $a_{k+1} = (g_{k+1}; i_{k+1}, j_{k+1}) \in S_y$, say, and $a_k = (g_k; i_k, j_k) \in S_x$, say. Thus, $a_{k+1}\delta = g_{k+1} \in T_y$ and $a_k\delta = g_k \in T_x$, and, hence, $g_{k+1} L g_k$ (in S/δ). Using Lemma 1.4, it easily seen that $y = z$ and $g_{k+1} = \mu g_k$ where μ is a unit of T_y. Since $a_{k+1} < a_k$, $a_{k+1} = s a_k$ for some $s \in S$. We may take $s = (s'; m, n) \in S_y$. Hence, $(g_{;k+1}\, i_{k+1}, j_{k+1}) = (s'; m, n)(g_k; i_k, j_k)$. So, $j_{k+1} = j_k$. Thus, $(g_k; i_k, j_k) = (\mu^{-1}; i_k, j_k) \cdot (g_{k+1}; i_{k+1}, j_{k+1})$. Hence, $a_{k+1} L a_k$, a contradiction. Thus, $\text{red}(a_n\delta, a_{n-1}\delta, \dots, a_1\delta) = (a_n\delta, a_{n-1}\delta, \dots, a_1\delta)$ and $|z| = |z\delta|$.

For $_t \in U = (\widehat{S/\delta})$, let $U_t = \{x \in U :_t x =_t\}$

Lemma 1.7. *For* $t \in U$, $U_t\widehat{\delta}^{-1} \leq E(\widehat{S})$. *If* $_t \in X_y$, $U_t\widehat{\delta}^{-1} \leq U(E(F_x) :_x \geq_y)$.

Proof. Let $s \in U_t\widehat{\delta}^{-1}$. Hence, $s\widehat{\delta} \in U_t$. Using an important theorem of Rhodes [13, Theorem A.1V.1], $(s\widehat{\delta})^{|t|+1} = (s\widehat{\delta})^{|t|}$. Let $s = (s_n, s_{n-1}, \dots, s_1)$. Then, $s\widehat{\delta} = (s_n\delta, s_{n-1}\delta, \dots, s_1\delta)$. If $s_n = (g; i, j) \in S_y$, $s_n\delta = g \in T_y$. Thus, $pr_1(s\widehat{\delta})^{|t|+1} = g^{|t|+1}$ and $pr_1(s\widehat{\delta})^{|t|} = g^{|t|}$. Let e denote the identity of T_y. Thus, since T_y is a cancellative monoid, $g^{|t|}e = g^{|t|}g$ implies $e = g$. Hence, $s_n \in E(S)$. Thus, using [24, Theorem 3.1(f)], $s \in E(\widehat{S})$. Hence $U_t\widehat{\delta}^{-1} \leq E(\widehat{S})$. The last sentence of the lemma is a consequence of the definitions of U_t and δ, Lemma 1.5, and the first sentence of the lemma.

If we replace "e" by "δ", "X_y" by "F_y", "G_y" by "T_y", and "U_y" by "X_y" in [26, Lemma 5, Lemma 7, Lemma 8, Lemma 9, Lemma 11] (if $U_t\widehat{\delta}^{-1} \neq \phi$ and the last sentence is omitted), Lemma 12, Lemma 13, the first two sentences of Lemma 15, Lemma 16, Lemma 17, and Lemma 18 (with "and $\cdots Y^x$" omitted)], these lemmas are valid for quasi-adequate semigroups S. The proofs of these modified lemmas are the same as the proofs of the original lemmas in [26] except that we replace Lemma 1 of [26] by Lemma 1.1, 1.4, and 1.5 and Proposition 1.3; Lemma 2 of [26] by Lemma 1.6; and Lemma 6 of

[26] by Lemma 1.7 in the proofs of the original lemmas. Using Lemmas 1.1, 1.4 and 1.5, Proposition 1.3, Lemma 1.6, [26, Lemma 3], Lemma 1.7, and the modified Lemmas, we obtain

Lemma 1.8. *If $U_t\widehat{\delta}^{-1} \neq \phi$, then $U_t\widehat{\delta}^{-1}$ is a chain $\widetilde{P}_{|t|}$ of rectangular bands $(W_j : j \in \widetilde{P}_{|t|})$ where $\widetilde{P}_{|t|}$ is a sub-chain of $P_{|t|} = \{1, 2, \cdots, |t|\}$ under the reverse of the usual order. Furthermore, every element of W_j has length j.*

Let $t \in X_y$ and suppose that $|t| = k$. If $x, y \in U_t\widehat{\delta}^{-1}$, define $x\sigma' y$ if and only if $ax = ay$ for all $a \in W_k$ where k is thé least element of $\widetilde{P}_k$.

If we make the usual modifications and furthermore replace "σ" by "σ'", [26, Lemma 21 and Lemma 23] are valid for super quasi-adequate semigroups S. The proofs also remain valid of we replace "σ" by "σ'", "e" by "δ", k by $\overline{k}$, and Lemma 7 by modified Lemma 7 if we note that $e_j L g_j$ (notation of [26, Lemma 23]) by virtue of the modified Lemma 5.

Lemma 1.9. *If $U_t\widehat{\delta}^{-1} \neq \phi$, L is a congruence relation on $U_t\widehat{\delta}^{-1}$. Hence, $U_t\delta^{-1}/L$ is a chain $\widetilde{P}_{|t|}$ of right zero semigroups $(W_j/L : j \in \widetilde{P}_{|t|})$.*

Proof. Replace "δ" for "e", Lemmas 21 and 23 by their modifications, and Lemma 1.8 for Lemma 20 in the proof of [26, Lemma 24].

Let r be a homomorphism of a monoid S onto a monoid T, we define a category R_r as follows: obj $R_r = T$. For $t_1, t_2 \in T$, $R_r(t_1, t_2) = \{(t_1, s, t_2) : s \in S \text{ and } t_2 = t_1(s_r)\}$. For $(t_1, s_1, t_2) \in R_r(t_1, t_2)$ and $(t_2, s_2, t_3) \in R_r(t_2, t_3)$, we define the composition $(t_1, s_1, t_2)(t_2, s_2, t_3) = (t_1, s_1s_2, t_3)$. It is easily checked that $(t_1, s_1s_2, t_3) \in R_r(t_1, t_3)$ and the composition is associative where defined. The identity arrow of $R_r(t,t)$ is $(t, 1, t)$ where 1 is the identity of S. So, R_r is a category. Let α be a congruence on S and for (t_1, s_1, t_2), $(t_1, s_2, t_2) \in R_r(t_1, t_2)$ define $(t_1, s_1, t_2)\Omega(t_1, s_2, t_2)$ if and only if $ss_1 = ss_2$ for all $s \in t_1^{r-1}$ and $s_1 \alpha s_2$. Then, by [26, Lemma 25], Ω is a congruence on the category R_r. Let $D_r^{\alpha} = R_r/\Omega$. Following Tilson [18], we term D_r^{α} the derived category of r. Let $[t_1, s_1, t_2] \in D_r^{\alpha}(t_1, t_2)$ denote the Ω-class of R_r containing $(t_1, s_1, t_2) \in R_r(t_1, t_2)$. We define $x\lambda y$ (in $\widehat{S}$) if $x, y \in F_v$ for some v. Clearly, λ is a congruence relation on $\widehat{S}$.

Lemma 1.10. *For $t \in (\widehat{S/\delta})$, $[t, s, t]\tau = sL$ defines an isomorphism of $D_{\widehat{\delta}}^{\lambda}(t, t)$ onto $(U_t\widehat{\delta}^{-1}/L)^1$.*

Proof. Suppose $sLz(s, z \in U_t\widehat{\delta}^{-1})$ Hence, using Lemma 1.8, $s, z \in W_j$ for some $j \in \widetilde{P}_{|t|}$. Thus, using modified [26, Lemma 23], $s\sigma' z$. Hence, $xs = xz$ for all $x \in W_{\overline{k}}$ where $k = |t|$. Since $t(x\widehat{\delta}) = t$, $t \leq x\widehat{\delta}$. Let $t = (g_k, g_{k-1}, \ldots, g_1)$. If $\overline{k} = k$, using [16, Proposition 7.1] (valid for arbitrary semigroups) (see also [17, Proposition 12.1]), Lemmas 1.6-1.8, and [24, Lemma 3.1[f], $x\widehat{\delta} = (e_k, g_{k-1}, \ldots, g_1)$ where $e_k^2 = e_k L g_k$. Using Lemmas 1.5, 1.6 and 1.8 if $u \in_t \widehat{\delta}^{-1}$, then $u = ((g_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_1; i_1, j_1))$, say. Since

$W_{\overline{k}} = W_k = E(F_{k'}) \cap U_t\widehat{\delta}^{-1}$ (where $_k \longrightarrow_{k'}$ defines isomorphism of $\widetilde{P}_k$ into Y) (see [26]), let $x = ((e_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_1; i_1, j_1))$. Since $(g_k; i_k, j_k)L(e_k; i_k, j_k)$, it is easily checked that $ux = u$. Hence, $us = uxs = uxz = uz$. Since $s, z \in W_j$, $s\lambda z$. Thus, $[_t, s_{,t}] = [_t, z_{,t}]$. Next, assume $_k >_{\overline{k}}$. Then, using [17, Proposition 12.1]), Lemma 1.7 and [24, Theorem 3.1(f)], $t = (g_k, g_{k-1}, \ldots, g_{\overline{k}}, g_{\overline{k}-1}, \ldots, g_1)$ and $x\widehat{\delta} = (e_{\overline{k}}, g_{\overline{k}-1}, \ldots, g_1)$ where $g_{\overline{k}} L e_{\overline{k}} = e^2_{\overline{k}}$. Hence, $u = (g_k; i_k, j_k), (g_{k-1}; i_{k-1}, j_{k-1}), \ldots, (g_{\overline{k}}, i_{\overline{k}}, j_{\overline{k}}), (g_{\overline{k}-1}; i_{\overline{k}-1}, j_{\overline{k}-1}), \ldots, (g_1; i_1, j_1))$ and $x = ((e_{\overline{k}}; i_{\overline{k}}, j_{\overline{k}}), (g_{\overline{k}-1}; i_{\overline{k}-1}, j_{\overline{k}-1}), \ldots, (g_1; i_1, j_1))$.

Since $(g_s; i_s, j_s) < (g_{\overline{k}}; i_{\overline{k}}, j_{\overline{k}})$ for $_{\overline{k}} <_s \leq_k$, $(g_s; i_s, j_s)(e_{\overline{k}}; i_{\overline{k}}, j_{\overline{k}}) = (g_s; i_s, j_s)$. Furthermore $(g_{\overline{k}}; i_{\overline{k}}, j_{\overline{k}})L(e_{\overline{k}}; i_{\overline{k}}, j_{\overline{k}})$. Hence, by a routine calculation, $ux = u$. Thus, as above, $[_t, s_{,t}] = [_t, z_{,t}]$. Conversely, assume $[_t, s_{,t}] = [_t, z_{,t}]$. Hence, $s, z \in F_q$, say and $xs = xz$ for all $x \in_t \widehat{\delta}^{-1}$. Using [26, Lemma 22], $s \leq z$ or $z \leq s$. Using Lemma 1.7, $sz = s$ or $zs = z$. Since $s, z \in W_j$ for some j, sLz in either case. Thus, $[_t, s_{,t}]\tau = sL(s \in U_t\widehat{\delta}^{-1})$ defines a 1-1 map of $D^{\lambda}_{\widehat{\delta}}(_{t,t})$ into $(U_t\widehat{\delta}^{-1}/L)^1$. Clearly, τ is a surjection. Using Lemma 1.9, τ is an isomorphism.

Theorem 1.11. *Let S be a super quasi-adequate semigroup. Then,*

$$S^1 < V \circ (\widehat{S/\delta})^1 \tag{1}$$

where V is an L-trivial and idempotent monoid, δ is the minimum adequate good congruence on S, $(\widehat{S/\delta})$ is a semilattice $Y = S/J^*$ of left cancellative semigroups $(X_y :_y \in Y)$ with idempotents, and $E((\widehat{S/\delta}))$ is a semilattice Y of right zero semigroups $(E(X_y) :_y \in Y)$.

Proof. Utilize Lemma 1.5 (define $1\delta = 1$), Lemma 1.10, [26, Lemma 29], and [26, Theorem 26] to establish (1). To complete the proof utilize Proposition 1.3 and Lemma 1.5.

Remark 1.12. If E is the edge set of the graph obtained from $D^{\lambda}_{\widehat{\delta}}$ by removing the identity arrows, then V is the free monoid over E relative to the equation $xyx = yx(x, y \in E^1)$ (see [26]-especially the proof of [26, Lemma 29]). V is a semilattice A (set of all finite subsets of E under union) of right zero semigroups $(U_\rho : P \in A)$ where U_ρ denotes the set of all elements of V with content P (see [2], [10] and [26, especially Theorem 27])

Lemma 1.12. *$X_y = C_y \times E_y$ where C_y is a cancellative monoid and E_y is a right zero semigroup if and only if T_y is a group. In the case, $X_y = T_y \times E(X_y)$.*

Proof. Suppose $X_y = C_y \times E_y$. Then, Using [19, Theorem 2], $a \in aX_y$ for all $a \in X_y$. Thus, $(a_n) = (a_n)e$ for some $e \in X_y$. Hence, $(a_n)e = (a_n)e^2$. Thus, using Lemma 1.5, $e = e^2$. Hence, using Lemma 1.5, $(a_n) = (a_n)(e_y, x_{k-1}, \ldots, x_1)$ where e_y is the identity of T_y. Thus, $(a_n) = \text{red}(a_n, e_y, x_{k-1}, \ldots, x_1)$. So, $a_n L e_y$. Hence, using Lemma 1.4, $e_y = sa_n$ where s may be taken as an element of T_y. Thus, $a_n s a_n s =$

$a_n e_y s = a_n s = a_n s e_y$. So, $a_n s = e_y$ and, hence, T_y is a group. Conversely, suppose T_y is a group. Let $(a_n, a_{n-1}, \ldots, a_1) \in X_y$. Then, $(a_n, a_{n-1}, \ldots, a_1) = (a_n)(e_y, a_{n-1}, \ldots, a_1)$. Since $(a_n)(b_n) = (a_n b_n)$ for $a_n, b_n \in T_y$, $T_y \cong \{(a_n : a_n \in T_y\}$. Thus, it is easily checked that every element of X_y may be uniquely expressed in the form $(a)e$ where $a \in T_y$ and $e \in E(X_y)$ and $(a, e) \longrightarrow (a)e$ defines an isomorphism of $T_y \times E(X_y)$ onto X_y.

Remark 1.13. In the case S is an orthodox union of groups in Theorem 1.11, δ becomes the minimum inverse semigroup congruence on S, $J^* = J$ and $X_y = T_y \times E(X_y)$ where T_y is a maximal subgroup of S (hence, X_y is a right group). These facts are a consequence of Proposition 1.3. Lemma 1.1, and Lemma 1.12. In this case, the structure of $(\widehat{S/\delta})$ is further refined by [25, Theorem 2.6] (see also [26, Theorem 31]).

Section 2. The Structure of Super Generalized L^*-unipotent Semigroups.

In this section, we describe the smallest L^*-unipotent good congruence on a super generalized L^*-unipotent semigroup (Proposition 2.1) and give a structure theorem for super generalized L^*-unipotent semigroups (Theorem 2.4).

Proposition 2.1. *Let S be a super generalized L^*-unipotent semigroup. Then, $\delta \cap L$ is the smallest L^*-unipotent good congruence on S. $S/\delta \cap L$ is a semilattice $Y = S/J^*$ of semigroups $(M_y :_y\in Y)$ where $M_y = T_y \times J_y$ where T_y is the cancellative monoid of Lemma 1.1 and J_y is an R-class of $E(S_y)$. $E(S/\delta \cap L)$ is a semilattice Y of the right zero semigroups $(J_y :_y\in Y)$.*

Proof. We first show that $\delta \cap L$ is a congruence relation on S. Utilizing Proposition 1.3, $\delta \cap L$ is a right congruence relation on S. Let $\overline{\delta \cap L}$ be the smallest congruence relation on S containing $\delta \cap L$. We will show that $\overline{\delta \cap L} = \delta \cap L$. Suppose a $(\overline{\delta \cap L})b$. Then, there exists $a = a_1, a_2, \ldots, a_n = b \in S$ such that $a_i = x_i u_i$, $a_{i+1} = x_i v_i$ where $x_i \in S^1$ and $(u_i, v_i) \in \delta \cap L$ for $1 \leq i \leq n-1$. Let $x_i = (g; i, k)_\gamma \in S_\gamma$, $u_i = (w; s, j)_\lambda \in S_\lambda$, and $v_i = (w; t, j)_\lambda \in S_\lambda$. Since δ is a congruence relation, $a_i = (m; p, q)_{\gamma\lambda}$ and $a_{i+1} = (m; i, d)_{\gamma\lambda}$, say. Let $_\alpha =_{\gamma\lambda}$. Then, $_{\alpha\gamma} = {}_{\alpha\lambda} = {}_\alpha$. Hence, $(m; p, q)_\alpha = (g; i, k)_\gamma (e_\gamma; i, k)_\gamma (e_\lambda; s, j)_\lambda (w; s, j)_\lambda$ and $(m; c, d)_\alpha = (g; i, k)_\gamma (e_\gamma; i, k)_\gamma (e_\lambda; t, j)_\lambda (w; s, j)_\lambda$ where e_γ is the identity of T_γ.

Since L is a congruence relation on $E(S)$, $(e_\gamma; i, k)_\gamma (e_\lambda; s, j)_\lambda L (e_\gamma; i, k)_\gamma (e_\lambda; t, j)_\lambda$. Hence, $(e_\lambda; i, k)_\gamma (e_\lambda; s, j)_\lambda = (e_\alpha; s', j')_\alpha$ and $(e_\gamma; i, k)_\gamma (e_\lambda; t, j)_\lambda = (e_\alpha; t', j')_\alpha$, say. Hence,

$$\begin{aligned}(m; p, q)_\alpha &= (g; i, k)_\gamma (e_\alpha; s', j')_\alpha (w; s, j)_\lambda \\ (m; c, d)_\alpha &= (g; i, k)_\gamma (e_\alpha; t', j')_\alpha (w; s, j)_\lambda\end{aligned}$$

Since L is a right congruence relation on S, $(e_\alpha; s', j')_\alpha (w; s, j)_\lambda L (e_\alpha; t', j')_\alpha (w; s, j)_\lambda$. Hence, $(e_\alpha; s', j')_\alpha (w; s, j)_\lambda = (w^*; s^*, j^*)_\alpha$ and $(e_\alpha; t', j')_\alpha (w; s, j)_\lambda = (\overline{w}; \overline{s}, j^*)_\alpha$, say. Thus,

$$\begin{aligned}(m;p,q)_\alpha &= (g;i,k)_\gamma(w^*;s^*,j^*)_\alpha\\ (m;c,d)_\alpha &= (g;i,k)_\gamma(w;s,j^*)_\alpha\end{aligned}$$

Hence,

$$\begin{aligned}(e_\alpha;p,q)_\alpha(m;p,q)_\alpha &= (e_\alpha;p,q)_\alpha(g;i,k)_\gamma(w^*;s^*,j^*)_\alpha\\ (e_\alpha;p,q)_\alpha(m;c,d)_\alpha &= (e_\alpha;p,q)_\alpha(g;i,k)_\gamma(\overline{w};\overline{s},j^*)_\alpha.\end{aligned}$$

Suppose that $(e_\alpha;p,q)_\alpha(g;i,k)_\gamma = (\overline{g};\overline{i},\overline{k})_\alpha$. Then,

$$\begin{aligned}(m;p,q)_\alpha &= (\overline{g};\overline{i},\overline{k})_\alpha(w^*;s^*,j^*)_\alpha\\ (m;p,d)_\alpha &= (\overline{g};\overline{i},\overline{k})_\alpha(\overline{w};\overline{s},j^*)_\alpha\end{aligned}$$

Hence, $q = d = j^*$. Thus, $a_i(\delta\cap L)a_{i+1}$ for $1 \le i \le n-1$. Hence, $a(\delta\cap L)b$ and, thus $\delta\cap L = \overline{\delta\cap L}$.

We will need to show that $\delta\cap L^* = \delta\cap L$. Suppose $a(\delta\cap L^*)b$. Since $a\delta b$, $a = (g;i,j)_\alpha \in S_\alpha$ and $b = (g;r,s)_\alpha \in S_\alpha$, say. There exists an oversemigroup S^* of S such that $s(g;i,j)_\alpha = (g;r,s)_\alpha$ where $s\in S^*$. Hence, $(g;r,s)_\alpha(e_\alpha;i,j)_\alpha = (g;r,s)_\alpha$. Thus, $j = s$. Hence, $a(\delta\cap L)b$. Thus, $\delta\cap L^* \le \delta\cap L$. Since $L\le L^*$, $\delta\cap L^* = \delta\cap L$.

We next show that $\delta\cap L$ is a good congruence. We will use [5, Corollary 1.5]. Suppose aL^*e where $e\in E(S)$. Let $ax(\delta\cap L)ay$ where $x,y\in S^1$. Thus, $ax(\delta\cap L^*)ay$. Since aL^*e, axL^*ex and ayL^*ey. Thus, exL^*ey. Using [5, Corollary 1.5] and Proposition 1.3, $ex\delta ey$ for some $e^2 = e\in L^*a$. Thus, $ex(\delta\cap L^*)ey$. Hence, $ex(\delta\cap L)ey$. Next, let aR^*e where $e\in E(S)$. Assume $xa(\delta\cap L)ya$ where $x,y\in S^1$. Thus, $xa = (h;m,n)_\alpha$ and $ya = (h;p,n)_\alpha$, say. Let $f = (e_y;m,n)_\alpha$. Then, $xa = fya$. Hence, $fxa = fya$. Thus, using [11, Lemma 1.7], $fxe = fye$. Since xaR^*xe and yaR^*ye, it is easily seen that xe, ye, and $f\in S_\alpha$. Hence $fxe = fye$ implies $xe(\delta\cap L)ye$. Thus, $\delta\cap L$ is a good congruence on S by [5, Corollary 1.5].

We next show that $S/\delta\cap L$ is an L^*-unipotent semigroup. Using [6, Proposition 1.6], $S/\delta\cap L$ is a quasi-adequate semigroup. Using [6, Lemma 1.5], $E(S/\delta\cap L) = \{e(\delta\cap L) : e\in E(S)\}$. Suppose $e(\delta\cap L)Lf(\delta\cap L)$ (in $E(S/\delta\cap L)$). Thus, $(ef,e)\in\delta\cap L$ and $(fe,f)\in\delta\cap L$. Hence, $e,f\in S_\gamma$, say. Thus, $e = efe = ef$. Hence, $e(\delta\cap L)f$. Thus, $S/\delta\cap L$ is an L^*-unipotent semigroup.

Let e be an L^*-unipotent congruence on S. Suppose $a(\delta\cap L)b$. Then, $a = (g;m,n)_\alpha$ and $b = (g;p,n)_\alpha$, say. Thus $a = (e_\alpha;m,n)_\alpha b$. Since $(e_\alpha;m,n)_\alpha L(e_\alpha;p,n)_\alpha$, $(e_\alpha;m,n)_\alpha e = (e_\alpha;p,n)_\alpha e$. Hence, $ae = (e_\alpha;m,n)_\alpha ebe = (e_\alpha;p,n)_\alpha be = be$. Thus, $\delta\cap L\le e$. Thus, $\delta\cap L$ is the smallest L^*-unipotent congruence on S.

Using Lemma 1.1, $S_y = T_y\times I_y\times J_y$ (algebraic direct product) where I_y is a left zero semigroup and J_y is a right zero semigroup. Let $M_y = T_y\times J_y$ (algebraic direct product). Let $\overline{(g;i,j)}$ denote the $\delta\cap L$-class of S containing $(g;i,j)$. Then, $\overline{(g;i,j)}\lambda = (g,j)$ defines a 1-1 mapping of $S/\delta\cap L$ onto $M = U(M_y :_y\in Y)$. In a similar manner to the proof of Lemma 1.4, we may define a multiplication on M such that M is a semilattice Y of the semigroups $(M_y :_y\in Y)$ and $M \cong S/\delta \cap L$. The last sentence follows since $E(M)$ is a semigroup.

Remark 2.2 will be used in the proof of Theorem 2.4.

Remark 2.2. Let θ be a homomorphism of a semigroup S onto a semigroup T. Define $D(\theta) = \{(t, s, t(s\theta)) : t \in T^{\circ}; s \in S\}U\{O\}$ under the multiplication $(t_1, s_1, t_1(s_1\theta))(t_2, s_2, t_2(s_2\theta)) = (t_1, s_1s_2, t_1(s_1s_2)\theta)$ if $t_1(s_1\theta) = t_2$; O if $t_1(s_1\theta) \neq t_2$ and $O(t, s, t(s\theta)) = (t, s, t(s\theta))O = O.O = O.D(\theta)$ was termed the derived semigroup of θ by its inventor Bret Tilson (see [16] and [17]). Let ϕ be a mapping of $D(\theta)-\{O\}$ into a semigroup P. Following Rhodes [13, Definition A.I.2.1, p. 94], we term $\phi : D(\theta) - \{O\} \to P$ a parametrization of $D(\theta)$ if 1) ϕ is a partial homomorphism of $D(\theta)-\{O\}$ into P (i.e. if $x, y \in D(\theta)-\{O\}$ and $xy \neq 0$, then $x\theta y\theta = (xy)\theta$) 2) ϕ satisfies the embedding condition: $s_1\theta = s_2\theta$ and $(t, s_1, t(s_1\theta))\phi = (t, s_2, t(s_2\theta))\phi$ for all $t \in T^*$ implies $s_1 = s_2$. For brevity, we also term P a parametrization of $D(\theta)$. Using [13, Proposition AI.2.3], $S \leq PoT$ where $p|S = \theta$ (p is the projection if PoT onto T). Following Rhodes [13], we define $D^R(\theta)$ (dual derived semigroup) as follows: $D^R(\theta) = (((s\theta)t, s, t) : s \in S, t \in T^{\circ})U\{o\}$ under the multiplication $((s_1\theta)t_1, s_1, t_1)((s_2\theta)t_2, s_2, t_2) = ((s_1\theta)t_1, s_1s_2, t_2)$ if $t_1 = (s_2\theta)t_2$; o if $t_1 \neq (s_2\theta)t_2; o((s\theta)t, s, t) = ((s\theta)t, s, t)o = oo = o$. A parametrization P^R of $D^R(\theta)$ is defined as above and $S \leq T\ o\ P^R$ with $p|S = \theta$.

Remark 2.3 will be needed for the statement of Theorem 2.4

Remark 2.3. Let W be a partial groupoid which is a union of a collection of pairwise disjoint subsemigroups $(T_y :_y\in Y)$ where Y is a semilattice. If $a \in T_y$, $b \in T_x$ and $y \geq x$ (in Y) imply ab is defined (in W) and $ab \in T_x$ and $x \geq w$ and $c \in T_w$ imply $(ab)c = a(bc)$, we term W a lower partial chain Y of the semigroups $(T_y :_y\in Y)$. Let X be a semilattice Y of semigroups $(X_y :_y\in Y)$ and let R and S be semigroups. For the definition of $WoXoR$ and $S \leq WoXoR$, see [24, p. 188 and p. 189].

Theorem 2.4. *Let S be a super generalized L^*-unipotent semigroup. Then,*
(1) $S \leq W^1o(E(S)/L)^1o(S/\delta \cap L)^1$ where W is a lower partial chain $Y = S/J^*$ of left zero subsemigroups of $E(S)$, $E(S)/L$ is a semilattice Y of right zero semigroups, and $\delta \cap L$ is the smallest L^*-unipotent good congruence on S. Furthermore,
(2) $S/\delta \cap L \leq (S/\delta \cap L/e)^1o(E(S)/L)^1$ where e is the smallest adequate good congruence on $S/\delta\cap L$ and $S/\delta\cap L/e$ is a strong semilattice Y of cancellative monoids $(T_y :_y\in Y)$ (T_y is a cancellative subsemigroup of S).

Proof. We will first establish that $S \leq (E(S))^1o(S/\delta \cap L)^1$. For each $(g, j)_y \in M_y$ ($_y\in Y$) (Notation of Proposition 2.1), select a representative element $u_{(g,j)y}$ in S_y. We first show that every element of S may be uniquely expressed in the form $w_{(ey,j)y}u_{(g,j)y}$ where $w_{(ey,j)y} \in (e_y, j)_y(\delta \cap L)^{-1}$. Let $(g; i, j)_y \in S_y$ and suppose $u_{(g,j)y} = (g; i_o, j)_y$. Then, $(g; i, j)_y = (e_y; i, j)_y(g; i_o, j)_y$ where $(e_y; i, j)_y \in (e_y, j)_y(\delta \cap L)^{-1}$. It is easily checked that the above expression is unique. If $_s = (g, j)_y$, let $^+_s = (e_y, j)_y$. Thus every element of S may be uniquely expressed in the form $w^+_s u_s$ where $w^+_s \in^+_s (\delta \cap L)^{-1}$.

Let ${}_t \in S/\delta \cap L$ and ${}_s \in_x (\delta \cap L)^{-1}$. Hence, we may write $u_t^s = f({}_{t,s})u_{tx}$ where $f({}_{t,s}) \in ({}_{tx})^+(\delta \cap L)^{-1}$. First assume S has an identity. For $({}_{t,s,t}({}_s(\delta \cap L))) \in D(\delta \cap L) - \{O\}$, define $({}_{t,s,t}({}_s(\delta \cap L)))\theta = f({}_{t,s})$. We will show that $\theta : D(\delta \cap L) - \{O\} \to E(S)$ is a parametrization of $D(\delta \cap L)$. It is easily checked that θ defines a mapping of $D(\delta \cap L) - \{O\}$ into $E(S)$. Next, we show that θ defines a partial homomorphism. Let $({}_{t_1,s_1,t_1}({}_{s_1}(\delta \cap L)))$, $({}_{t_2,s_2,t_2}({}_{s_2}\delta \cap L))) \in D(\delta \cap L)$ with ${}_{t_1}({}_{s_1}(\delta \cap L)) ={}_{t_2}$. We must show $f({}_{t_1,s_1})f({}_{t_2,s_2}) = f({}_{t_1,s_1,s_2})$. Suppose ${}_{s_1} \in_{x_1} (\delta \cap L)^{-1}$ and ${}_{s_2} \in_{x_2} (\delta \cap L)^{-1}$. Then, $u_{t_1}(s_1, s_2) = f({}_{t_1,s_1s_2})u_{t_1x_1x_2} = f({}_{t_1,s_1s_2})u_{t_2x_2}$ where $f({}_{t_1,s_1s_2}) \in ({}_{t_2x_2})^+(\delta \cap L)^{-1}$. However, $(u_{t_1}s_1)s_2 = f({}_{t_1,s_1})(u_{t_2}s_2) = f({}_{t_1s_1})f({}_{t_2,s_2})u_{t_2x_2}$. Let ${}_{t_2} \in M_y$ and ${}_{x_2} \in M_x$, say. Hence, ${}_{t_2s_2} \in M_{yx}$. Furthermore, $\overset{+}{{}_{t_2}} \in E(M_y)$ and $({}_{t_2s_2})^+ \in E(M_{yx})$. Using the last sentence of Proposition 2.1, $\overset{+}{{}_{t_2}}({}_{t_2x_2})^+ = (\overset{+}{{}_{t_2}}({}_{t_2x_2})^+)({}_{t_2x_2})^+ = ({}_{t_2x_2})^+$. Hence, $f({}_{t_1,s_1})f({}_{t_2,s_2}) \in ({}_{t_2x_2})^+(\delta \cap L)^{-1}$. Thus, $f({}_{t_1,s_1})f({}_{t_2,s_2}) = f({}_{t_1,s_1s_2})$, and, hence, θ is a partial homomorphism. We next show the embedding condition is valid. Let e denote the identity of $S/\delta \cap L$ and let $u_e = 1$, the identity of S. Thus, if ${}_{s_1}(\delta \cap L) ={}_{s_2} (\delta \cap L) ={}_x$ and $f({}_{e,s_1}) = f({}_{e,s_2})$, then ${}_{s_1} = u_e s_1 = f({}_{e,s_1})u_x = f({}_{e,s_2})u_x = u_e s_2 ={}_{s_2}$. Hence, $E(S)$ is a parametrization of $D(\delta \cap L)$. Thus, using Remark 2.2, $S \leq E(S) o S/\delta \cap L$. If S has no identity consider S^1. Note that $a(\delta \cap L)_1$ (in S^1) implies $a ={}_1$. Hence, $S^1/\delta \cap L \cong (S/\delta \cap L)^1$. Furthermore, $E(S^1) \cong (E(S))^1$. Hence, $S \leq S^1 \leq (E(S))^1 o (S/\delta \cap L)^1$. Thus utilizing [24, Theorem 1.24, Remark (1.24)$'$, Lemma 1.23, and Lemma 1.25], we obtain (1). We next establish (2). Let $M = S/\delta \cap L$. Utilizing [9, Corollary 6.2 and Proposition 6.5], Proposition 2.1 and Lemma 1.4, M/e is the strong semilattice Y of cancellative monoids $(T_y :_y\in Y)$. If ${}_s \in T_y$, let ${}_s^* = e_y$, the identity of T_y. For each ${}_s \in M/e$, select a representative element $u_s \in_s e^{-1}$. We show that every element of M may be uniquely expressed in the form $u_s w_s^*$ where $w_s^* \in_s^* e^{-1}$. Let $(g, j)_y \in M_y$ and suppose $u_s = (g, j_0)_y \in M_y$. Hence, $(g, j)_y = (g, j_0)_y(e_y, j)_y$ where $(e_y, j)_y \in e_y e^{-1}$ and $g^* = e_y$. Suppose $u_s g_s^* = u_s h_s^*$. Then, since $M_y(_y\in Y)$ is left cancellative, $g_s^* = h_s^*$. Let ${}_t \in M/e$ and ${}_s \in_x e^{-1}$. Hence, we may write ${}_s u_t = u_{xt} f({}_{s,t})$ where $f({}_{s,t}) \in ({}_{xt})^* e^{-1}$. First, assume that M has an identity. For $(({}_s e)_{t,s,t}) \in D^R(e) - \{O\}$, define $(({}_s e)_{t,s,t})\theta = f({}_{s,t})$. Using the fact that M/e is a strong semilattice Y of cancellative monoids $(T_y :_y\in Y)$, we proceed as above to show that $\theta : D^R(e) - \{O\} \to E(M)$ is a parametrization of $D^R(e)$. Thus, using Remark 2.2, $M \leq M/e \overset{\Omega}{o} E(M)$. Again, proceeding as above, $M \leq M^1 \leq (M/e)^1 \overset{\Omega}{o} (E(M))^1$. Using Proposition 2.1, $E(M) \cong E(S)/L$. Hence (2) is valid. To complete the proof, utilize Proposition 2.1.

Remark 2.5. *W is a lower partial chain Y of L-classes of $E(S)$. Each J-class of $E(S)$ contains precisely one of these L-classes (see [24, Theorem 1.24]).*

Remark 2.6. Let S be a generalized L-unipotent union of groups. Then, $\delta \cap L$ is the smallest L-unipotent congruence on S (δ is the smallest inverse semigroup congruence on S), e is the smallest inverse semigroup congruence on $S/\delta \cap L$, T_y is a maximal subgroup of S, and $J^* = J$ in the statement of Theorem 2.4. Thus, Theorem 2.4 generalizes [24, Theorem 1.27, Theorem 1.28, and Theorem 1.26] in the case S is also a union of groups (our structure theorem for generalized L-unipotent unions of groups). A different type structure theorem for generalized R-unipotent unions of groups is given in [22, Theorem 4.7].

Section 3 Super R^*-unipotent Semigroups

In this section, we give a structure theorem for super R^*-unipotent semigroups (Theorem 3.1)

Theorem 3.1. *Let S be a super R^*-unipotent semigroup. Thus, $* \; S \leq (E(S))^1 o \, (S/\delta)^1$ where $E(S)$ is a semilattice $Y = S/J^*$ of left zero semigroups, δ is the smallest adequate good congruence on S, and S/δ is a strong semilattice Y of cancellative monoids $(T_y :_y \in Y)$ (T_y is a subsemigroup of S).*

Proof. Using Lemma 1.1, $S_y = T_y \times E(S_y)$ where $E(S_y)$ is a left zero semigroup. Hence, by a routine calculation, $\delta \cap L = \delta$. Thus, utilizing the proof of Theorem 2.4, $*$ is valid. Use Proposition 1.3 and Lemma 1.4 to complete the proof.

Remark 3.2. Let S be an R-unipotent union of groups. Then, δ is smallest inverse semigroup congruence on S, T_y is a maximal subgroup of S, and $J = J^*$ in the statement of Theorem 3.1. Hence, Theorem 3.1 generalizes [24, Remark 1.14, Theorem 1.12, and Theorem 1.8] (our structure theorem for R-unipotent unions of groups). A different type structure theorem for L-unipotent unions of groups is given in [22, Theorem 7.2].

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