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ON COMPANION INEQUALITIES RELATED TO HEINIG'S

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Abstract. A number of companion inequalities related to Heinig's and their discrete analogues are investigated.

I. Introduction.

In [1], Heinig established the following three inequalities:

Theorem A. *Let p, s, λ be real numbers satisfying $p+s>\lambda,\ p>0$. If*

$$\int_0^\infty t^{\lambda-s} \mid f(t) \mid dt \ < \ \infty,$$

then,

$$\int_0^\infty x^\lambda \exp\big[px^{-p}\int_0^x t^{p-1}\log \mid x^{-s}f(t) \mid dt\big]dx \ \le \ e^{1/p}A\int_0^\infty t^{\lambda-s} \mid f(t) \mid dt, \quad (1)$$

where $A=p/(p+s-\lambda)$.

Theorem B. *Let $2p-1>\lambda-sp,\ p>0$ and*

$$\int_0^\infty t^{\lambda-sp} \mid f(t) \mid^p dt \ < \ \infty.$$

Then

$$\int_0^\infty x^\lambda \exp\big[p^2x^{-p}\int_0^x t^{p-1}\log \mid x^{-s}f(t) \mid dt\big]dx \ \le \ e\,B\int_0^\infty t^{\lambda-sp} \mid f(t) \mid^p dt, \quad (2)$$

where $B=p/(2p+sp-\lambda-1)$.

Theorem C. *Suppose $\{a_n\}_{n=1}^\infty$ is a non-negative sequence and $s>0,\ p\ge 1,\ 0\le \lambda<s+p$, If $\sum_{n=1}^\infty n^{\lambda-s}a_n=M<\infty$,*

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then

$$\sum_{n=1}^{\infty} n^{\lambda-sp} \prod(a_n,p) \ \le \ e^{1/p} C \sum_{n=1}^{\infty} n^{\lambda-s} a_n, \tag{3}$$

where $\prod(a_n,p) = [a_1 a_2^{2^{p-1}} \cdots a_n^{n^{p-1}}]^{p/n^p}$, $n = 1,2,3,\cdots$, *and* $C = p[1+1/(p+s-\lambda)]$.

The following improvement of the inequalities (1), (2) and (3) are given by Cochran and Lee [3].

Theorem D. *Let p, r be real numbers with $p > 0$. If*

$$\int_0^{\infty} t^r \mid f(t) \mid dt \ < \ \infty$$

then

$$\int_0^{\infty} x^r \exp\big[px^{-p} \int_0^{x} t^{p-1} \log \mid f(t) \mid dt\big] dx \ \le \ e^{(r+1)/p} \int_0^{\infty} t^r \mid f(t) \mid dt. \tag{4}$$

Theorem E. *Let p, r be real numbers with $p \ge 1$, $r \ge 0$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative real numbers less then or equal to unity with $\sum_{n=1}^{\infty} n^r a_n < \infty$, then*

$$\sum_{n=1}^{\infty} n^r \prod(a_n,p) \ \le \ e^{(r+1)/p} \sum_{n=1}^{\infty} n^r a_n. \tag{5}$$

The purpose of this paper is to establish some companion inequalities that are related to inequalities (1)-(5).

II. Companion inequalities related to Heinig's.

Theorem 1. *Let p, s, λ be real numbers satisfying $p + s < \lambda$, $p < 0$, If*

$$\int_0^{\infty} t^{\lambda-s} \mid f(t) \mid dt \ < \ \infty,$$

then

$$\int_0^{\infty} x^{\lambda} \exp\big[-px^{-p} \int_x^{\infty} t^{p-1} \log \mid x^{-s} f(t) \mid dt\big] dx \ \le \ e^{1/p} A \int_0^{\infty} t^{\lambda-s} \mid f(t) \mid dt, \tag{6}$$

where $A = p/(p+s-\lambda)$.

Proof. Since

$$e^{1/p} \ = \ \exp\big[p \int_1^{\infty} y^{p-1} \log \ y \ dy\big]$$

a change of variable shows that (6) has the form

$$\int_0^\infty x^\lambda \exp\big[-px^{-p}\int_1^\infty x^{p-1}y^{p-1}\log\mid x^{-s}f(xy)\mid xdy\big]\,dx$$
$$\le e^{1/p}A\int_0^\infty t^{\lambda-s}\mid f(t)\mid dt,$$

which is equivalent to

$$\int_0^\infty x^\lambda \exp\big[-p\int_1^\infty y^{p-1}(\log\mid x^{-s}yf(xy)\mid)dy\big]\,dx \;\le\; A\int_0^\infty t^{\lambda-s}\mid f(t)\mid dt.$$

But by Jensen's inequality ([5], p.62) the left side of the above inequality is dominated by

$$-p\int_0^\infty x^\lambda\big[\int_1^\infty \mid x^{-s}y^p f(xy)\mid dy\big]\,dx$$
$$= -p\int_1^\infty y^{p+s-\lambda-1}\big[\int_0^\infty t^{\lambda-s}\mid f(t)\mid dt\big]\,dy$$
$$= A\int_0^\infty t^{\lambda-s}\mid f(t)\mid dt$$

which is obtained by an interchange of order of integration which is justified by Fubini's Theorem.

Theorem 2. *Let* $2p-1<\lambda-sp,\ p<0$ *and*

$$\int_0^\infty t^{\lambda-sp}\mid f(t)\mid^p dt \;<\; \infty.$$

Then

$$\int_0^\infty x^\lambda \exp\big[-p^2x^{-p}\int_x^\infty t^{p-1}\log\mid x^{-s}f(t)\mid dt\big]\,dx$$
$$\le eB\int_0^\infty t^{\lambda-sp}\mid f(t)\mid^p dt, \tag{7}$$

where $B=p/(2p+sp-\lambda-1)$.

Proof. Since

$$e \;=\; \exp\big[p^2\int_1^\infty y^{p-1}\log\ y\ dy\big]$$

a change of variable shows that (7) has the form

$$\int_0^\infty x^\lambda \exp\big[-p^2x^{-p}\int_1^\infty x^{p-1}y^{p-1}\log\mid x^{-s}f(xy)\mid xdy\big]\,dx$$
$$\le eB\int_0^\infty t^{\lambda-sp}\mid f(t)\mid^p dt$$

which is equivalent to

$$\int_0^\infty x^\lambda \exp\left[-p^2 \int_1^\infty y^{p-1} \log \mid x^{-s} y f(xy) \mid dy\right] dx \;\le\; B \int_0^\infty t^{\lambda - sp} \mid f(t) \mid^p dt.$$

By Jensen's inequality the left side of the above inequality is dominted by

$$\begin{aligned}
&\int_0^\infty x^\lambda \Big[\int_1^\infty \mid x^{-s} y f(xy) \mid^p (-p) y^{p-1} dy\Big] dx \\
&= -p \int_1^\infty y^{2p-1} \Big[\int_0^\infty x^{\lambda - sp} \mid f(xy) \mid^p dx\Big] dy \\
&= B \int_0^\infty t^{\lambda - sp} \mid f(t) \mid^p dt,
\end{aligned}$$

which is obtained by an interchang of order of integration which is justified by Fubini's Theorem.

For the discrete analogue of theorem 1 it is convenient to introduce the following notation. Let $\{a_n\}_{n=1}^\infty$ be a sequence of non-negative real numbers and $p < 0$. Then we write

$$Q(a_n, p) \;=\; \left[a_n^{n^{p-1}} a_{n+1}^{(n+1)^{p-1}} \cdots\right]^{-p/n^p}, \; n \;=\; 1, 2, 3, \cdots.$$

Theorem 3. *Suppose $\{a_n\}_{n=1}^\infty$ is a non-negative sequence and $s \le 0$, $p \le -1$, $\lambda \ge 0$. If $\sum_{n=1}^\infty n^{\lambda - s} a_n = M < \infty$,*
then

$$\sum_{n=1}^\infty n^{\lambda - sp} Q(a_n, p) \;\le\; e^{1/p} C \sum_{n=1}^\infty n^{\lambda - s} a_n, \tag{8}$$

where $C = -p[1 + 1/(\lambda - p - s)]$.

Proof. Without loss of generality, we may assume that

$$n^{-s} a_k \;\le\; 1, \; k \;=\; n, n+1, \cdots.$$

If $0 < M \le 1$, this is obvious.
If $M > 1$, then divide both side of (8) by M to obtain

$$\frac{1}{M} \sum_{n=1}^\infty n^{\lambda - sp} Q\left(\frac{a_n}{M}, p\right) M^\alpha \;\le\; e^{1/p} C \sum_{n=1}^\infty n^{\lambda - s} \frac{a_n}{M}.$$

where $\alpha = (-p \sum_{k=n}^\infty k^{p-1})/n^p \ge 1 \ge p$.
It follows that

$$\frac{1}{M} \sum_{n=1}^\infty n^{\lambda - sp} Q\left(\frac{a_n}{M}, p\right) \;\le\; e^{1/p} C \sum_{n=1}^\infty n^{\lambda - s} \frac{a_n}{M}.$$

Replacing a_n/M by a_n, we obtain $n^{\lambda-s}a_n \le 1$ and hence $n^{-s}a_n \le n^{-\lambda} \le 1$ which implies $n^{-s}a_k \le 1,\ k = n, n+1, \cdots$.

Now to prove (8), observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\lambda-sp} Q(a_n,p)\\
&\le \sum_{n=1}^{\infty} n^{\lambda+\frac{sp}{n^p}\sum_{k=n}^{\infty}k^{p-1}} Q(a_n,p)\\
&= \sum_{n=1}^{\infty} n^{\lambda}\left[n^{-sn^{p-1}} n^{-s(n+1)^{p-1}}\cdots\right]^{\frac{-p}{n^p}} \left[a_n^{n^{p-1}} a_{n+1}^{(n+1)^{p-1}}\cdots\right]^{\frac{-p}{n^p}}\\
&= \sum_{n=1}^{\infty} n^{\lambda}\exp\left[\frac{-p}{n^p}\sum_{k=n}^{\infty}k^{p-1}\log(n^{-s}a_k)\right]\\
&= \sum_{n=1}^{\infty} n^{\lambda}\exp\left[\frac{-p}{n^p}\sum_{k=n}^{\infty}k^{p-1}\log(n^{-s}a_k)\right]\\
&= \sum_{n=1}^{\infty} n^{\lambda}\exp\left\{\frac{-p}{n^p}\sum_{k=n}^{\infty}k^{p-1}\int_k^{k+1}\log[n^{-s}f(t)]\,dt\right\},
\end{aligned}$$

where

$$f(t) = \begin{cases} a_k, & k < t \le k+1,\\ 0, & \text{otherwise.}\end{cases}$$

But since $n^{-s}f(t) \le 1$ the last equality is dominated by

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\lambda}\exp\left\{\frac{-p}{n^p}\sum_{k=n}^{\infty}\int_k^{k+1} t^{p-1}\log[n^{-s}f(t)]\,dt\right\}\\
&= \sum_{n=1}^{\infty} n^{\lambda}\exp\left\{\frac{-p}{n^p}\int_n^{\infty} t^{p-1}\log[n^{-s}f(t)]\,dt\right\}\\
&= \sum_{n=1}^{\infty} n^{\lambda}\exp\left\{-p\int_1^{\infty} y^{p-1}\log[n^{-s}f(ny)]\,dy\right\},
\end{aligned}$$

which is less than or equal to $Ce^{1/p}\sum_{n=1}^{\infty} n^{\lambda-s}a_n$ if , and only if

$$\sum_{n=1}^{\infty} n^{\lambda}\exp\left\{-p\int_1^{\infty} y^{p-1}\log[n^{-s}yf(ny)]\,dy\right\} \le C\sum_{n=1}^{\infty} n^{\lambda-s}a_n. \tag{9}$$

By Jensen's inequality, the left side of (9) is dominated by

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\lambda}[-p\int_1^{\infty} y^p n^{-s} f(ny)dy] \\
&= \sum_{n=1}^{\infty} n^{\lambda}[-p\int_n^{\infty} t^p n^{-p-s-1} f(t)dt] \\
&= -p\sum_{n=1}^{\infty} n^{\lambda-p-s-1}\sum_{k=n}^{\infty} a_k \int_k^{k+1} t^p\,dt \\
&\leq -p\sum_{n=1}^{\infty} n^{\lambda-p-s-1}\sum_{k=n}^{\infty} a_k k^p \\
&= -p\sum_{k=1}^{\infty} k^p a_k \sum_{n=1}^{\infty} n^{\lambda-s-p-1} \\
&= -p\sum_{k=1}^{\infty} k^p a_k [\sum_{n=1}^{k-1} n^{\lambda-s-p-1} + k^{\lambda-s-p-1}] \\
&\leq -p\sum_{k=1}^{\infty} k^p a_k [\int_0^k t^{\lambda-s-p-1}dt + k^{\lambda-s-p-1}] \\
&= -p\sum_{k=1}^{\infty} k^p a_k [k^{\lambda-s-p}(\frac{1}{\lambda-s-p} + \frac{1}{k})] \\
&\leq -p\sum_{k=1}^{\infty} k^p a_k [k^{\lambda-s-p}(\frac{1}{\lambda-s-p} + 1)] \\
&= -p(1+\frac{1}{\lambda-s-p})\sum_{n=1}^{\infty} n^{\lambda-s} a_n. \\
&= C\sum_{n=1}^{\infty} n^{\lambda-s} a_n.
\end{aligned}
$$

This completes the proof of this theorem.

The following theorem has been proved in [4], here, we give a simple proof which is motivated in part after [3]:

Theorem 4. *Let p, β be real numbers with $p < 0$. If $\int_0^{\infty} t^{\beta}|f(t)|dt < \infty$ and $\int_0^{\infty} t^{p-1}\log|f(t)|dt < \infty$,*
then

$$\int_0^{\infty} x^{\beta}\exp[-px^{-p}\int_x^{\infty} t^{p-1}\log\mid f(t)\mid dt]dx. \leq \exp(\frac{\beta+1}{p})\int_0^{\infty} x^{\beta}\mid f(x)\mid dx \quad (10)$$

Proof. Consider the following reslt of Hardy ([2], p.246):
If $q > 1,\ r < 1$ and $F(x) = \int_x^\infty |g(t)|dt$, then

$$\int_0^\infty x^{-r}F^q dx \ < \ (\frac{q}{1-r})^q \int_0^\infty x^{-r}(x \mid g \mid)^q dx.$$

Let $\alpha = 1 - r > 0$. Then

$$\int_0^\infty x^{\alpha-1}[\int_x^\infty \mid g(t) \mid dt]^q dx \ < \ (\frac{q}{\alpha})^q \int_0^\infty x^{q+\alpha-1} \mid g \mid^q dx.$$

Let $g(x) = x^{p-1}f^{1/q}$ and set $\beta = pq + \alpha - 1$. Then

$$\int_0^\infty x^{\alpha-1}[-p\int_x^\infty t^{p-1} \mid f(t) \mid^{1/q} dt]^q dx \ < \ (\frac{-pq}{-pq+\beta+1})^q \int_0^\infty x^{q+\alpha-1}x^{pq-q} \mid f \mid dx$$

so that

$$\int_0^\infty x^{\beta}[-px^{-p}\int_x^\infty t^{p-1} \mid f(t) \mid^{1/q} dt]^q dx \ < \ (1 - \frac{\beta+1}{pq})^{-q} \int_0^\infty x^{\beta} \mid f \mid dx$$

The desired inequality is obtained by taking the limit $q \longrightarrow \infty$.

Remark 1. If we set $\lambda - s = \beta$, then the left-side of the inequality (6) becomes the left-side of the inequality (10). And the right-side of inequality (6) is equal to

$$e^{1/p}(\frac{p}{p-\beta}\int_0^\infty t^{\beta} \mid f(t) \mid dt$$

Since

$$e^{(\beta+1)/p} \ < \ e^{1/p}(\frac{p}{p-\beta})$$

for $\lambda - s = \beta > p$, except $\beta = 0$.

This shows that the inequality (6) can not be sharp in general and theorem 4 represents an improvement of theorem 1.

Remark 2. If we set $\lambda - sp = \beta$ and replaced $|f|^p$ by $|f|$, then the left-side of the inequality (7) becomes the left-side of the inequality (10) and the right-side of (7) is equal to

$$e[p/(2p - \beta - 1)] \int_0^\infty t^{\beta} \mid f(t) \mid dt$$

Since

$$e^{(\beta+1)/p} \ < \ e[p/(2p - \beta - 1)]$$

for $\beta > 2p - 1$, except $\beta = p - 1$.

This shows that the inequality (7) can not be sharp in general and theorem 4 represents an improvement of theorem 2.

Remark 3. When $f(t) = t^{b-\beta}e^{-t^{-p/2}}$, with $b > -1$, (10) gives rise to

$$e^{(\beta-b)/p}2^{2(b+1)/p}\int_0^\infty x^b e^{-x^{-p/2}}dx \le e^{(\beta+1)/p}\int_0^\infty x^b e^{-x^{-p/2}}dx.$$

By letting $b \longrightarrow -1$, we readily deduce that the multiplicative constant appearing on the right-hand side of (10) must be best possible.

To prove the finally theorem, which is the principle result in the discrete case, we need the following:

Lemma. *If $0 \le s < 1$ and $0 \le s+\alpha \le 1$, then*

$$[(i+1)^{s+\alpha} - i^{s+\alpha}]\sum_{n=1}^{i} n^{-s} \le i^\alpha \frac{s+\alpha}{1-s}, \quad i = 1,2,3,\cdots$$

Proof. Since

$$\sum_{n=1}^{i} n^{-s} \le \int_0^i x^{-s}dx = \frac{i^{1-s}}{1-s},$$

it follows from mean value theorem that

$$\begin{aligned}
&[(i+1)^{s+\alpha} - i^{s+\alpha}]\sum_{n=1}^{i} n^{-s}\\
&\le [(i+1)^{s+\alpha} - i^{s+\alpha}]\frac{i^{1-s}}{1-s}\\
&= (s+\alpha)C^{s+\alpha-1}\frac{i^{1-s}}{1-s}, \qquad i < C < i+1\\
&\le (s+\alpha)i^{s+\alpha-1}\frac{i^{1-s}}{1-s}\\
&= \frac{s+\alpha}{1-s}i^\alpha.
\end{aligned}$$

Theorem 5. *Let p, r be real numbers with $p \le -1$, $0 \le r \le 1$. If $\{a_n\}_{n=1}^\infty$ is a sequence such that $0 \le a_n \le 1$. $\forall n$ and $\sum_{n=1}^\infty n^r a_n < \infty$, then*

$$\sum_{n=1}^\infty n^r Q(a_n, p) \le (-p)e^{(r-p)/p}\sum_{n=1}^\infty n^r a_n, \tag{11}$$

where

$$Q(a_n, p) = [a_n^{n^{p-1}} a_{n+1}^{(n+1)^{p-1}} \cdots]^{-p/n^p}, \quad n = 1,2,3,\cdots$$

Proof. Our demonstration is modelled in part after [3]. Since $0 \le a_n \le 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^r Q(a_n, p) \\
&= \sum_{n=1}^{\infty} n^r \exp\Big[-pn^{-p} \sum_{i=n}^{\infty} i^{p-1} \log a_i\Big] \\
&= \sum_{n=1}^{\infty} n^r \exp\Big[-pn^{-p} \sum_{i=n}^{\infty} \int_i^{i+1} i^{p-1} \log f(t) dt\Big] \\
&\le \sum_{n=1}^{\infty} n^r \exp\Big[-pn^{-p} \sum_{i=n}^{\infty} \int_i^{i+1} t^{p-1} \log f(t) dt\Big] \\
&= \sum_{n=1}^{\infty} n^r \exp\Big[-pn^{-p} \int_n^{\infty} t^{p-1} \log f(t) dt\Big] \\
&= e^{(r-p+s)/p} \sum_{n=1}^{\infty} n^r \exp\Big[-p \int_1^{\infty} y^{p-1} \log f(ny) dy\Big] \\
&\quad \exp\Big[-p \int_1^{\infty} y^{p-1} \log y^{r-p+s} dy\Big] \\
&= e^{(r-p+s)/p} \sum_{n=1}^{\infty} n^r \exp\Big\{-p \int_1^{\infty} y^{p-1} \log\big[y^{r-p+s} f(ny)\big] dy\Big\},
\end{aligned}$$

where

$$f(t) = \begin{cases} a_i, & i < t \le i+1, \ i = n, n+1, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

and s is chosen so that $0 \le s < 1$, and $0 \le s + r \le 1$. By Jensen's inequality, the final summation is dominated by

$$I = -p \sum_{n=1}^{\infty} n^r \int_1^{\infty} f(ny) y^{r+s-1} dy$$

and for this equality, we have

$$\begin{aligned}
I &= -p\sum_{n=1}^{\infty} n^r \int_n^{\infty} f(t)t^{r+s-1}n^{-r-s+1}n^{-1}dt \\
&= -p\sum_{n=1}^{\infty} n^{-s} \int_n^{\infty} f(t)t^{r+s-1}dt \\
&= -p\sum_{n=1}^{\infty} n^{-s} \sum_{i=n}^{\infty} a_i \int_i^{i+1} t^{r+s-1}dt \\
&= \frac{-p}{r+s}\sum_{n=1}^{\infty} n^{-s} \sum_{i=n}^{\infty} a_i[(i+1)^{r+s} - i^{r+s}] \\
&= \frac{-p}{r+s}\sum_{i=n}^{\infty} a_i[(i+1)^{r+s} - i^{r+s}]\sum_{n=1}^{i} n^{-s} \\
&\leq \frac{-p}{r+s}\sum_{i=1}^{\infty} a_i i^r \frac{r+s}{1-s} \qquad \text{(by Lemma)} \\
&= \frac{-p}{1-s}\sum_{i=1}^{\infty} a_i i^r
\end{aligned}$$

for $0 \leq s < 1$, and $0 \leq s + r \leq 1$, by virtue of the definition of $f(t)$ and the previous lemma. The proof is completed by noting that

$$-pe^{(r-p)/p} = \min_s\left[\frac{-p}{1-s}e^{(r-p+s)/p}\right],$$

where the minimum occurs for $s = 0$.

Remark 4. Since

$$e^{(r-p)/p} \leq e^{1/p}\{1 + [1/(r-p)]\} \qquad \text{for} \qquad p \leq -1,\ 0 \leq r \leq 1.$$

If we set $r = \lambda - s$, then our inequality (11) of theorem 5 represents a substantial improvement over (8) of theorem 3 in case $\lambda - s \leq 1$.

References

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