TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 4, Winter 1991

A NOTE ON A COVERING THEOREM OF KY FAN

HWEI-MEI KO AND KOK-KEONG TAN

Abstract. A covering theorem by open sets is proved to be equivalent to a covering theorem by closed sets due to Ky Fan. Dual theorems of Shapley and Knaster et al. are obtained as applications.

Throughout this note, we shall denote by σ an n-dimensional simplex in a Euclidean space and $\mathcal{F}$ the family of all faces of σ.

The following covering theorem by closed sets was proved by Fan in [1, Theorem 13]:

Theorem 1. (Fan). *For each $\tau \in \mathcal{F}$, let $p(\tau)$ and $q(\tau)$ be two given points in σ and let $A(\tau)$, $B(\tau)$ be two closed subsets of σ such that*

(a) $\cup_{\tau\in\mathcal{F}} A(\tau) = \cup_{\tau\in\mathcal{F}} B(\tau) = \sigma$.

(b) For each $\tau \in \mathcal{F}$ of dimension $< n$ and for any point $x \in \tau$, there is a $\rho \in \mathcal{F}$ such that $x \in B(\rho)$ and $q(\rho) \in \tau$.

Then there exist two non-empty subfamilies $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ such that

(c) $\left[\cap_{\tau\in\mathcal{G}} A(\tau)\right] \cap \left[\cap_{\rho\in\mathcal{H}} B(\rho)\right] \neq \phi$

and

(d) the convex hull of $\{p(\tau) : \tau \in \mathcal{G}\}$ meets the convex hull of $\{q(\rho) : \rho \in \mathcal{H}\}$.

As is observed by Fan, the condition (b) in Theorem 1 above is implied by the combination of the following two conditions:

(b') For each $\tau \in \mathcal{F}$, $q(\tau)$ is a point in τ,

(b'') For each $\tau \in \mathcal{F}$ of dimension $< n$ and for any point $x \in \tau$, there is a $\rho \in \mathcal{F}$ such that $\rho \subset \tau$ and $x \in B(\rho)$,

and the less general result obtained by replacing (b) by (b') and (b'') in Theorem 1 includes Shapley's theorem [3] as a special case which in turn generalizes the classical theorem of Knaster-Kuratowski-Mazurkiewicz [2]. We remark here that Shapley's result plays an important rule in game theory.

Receive August 13, 1990.
1980 Mathematics Subject Classification: 52A20.
This work partially supported by NSERC of Canada under grant A-8096.

In this note we shall show that Fan's Theorem 1 above is equivalent to the following result, namely, Theorem 2, which is a covering theorem by open sets and is a dual to Theorem 1:

Theorem 2. *For each $\tau \in \mathcal{F}$, let $p(\tau)$ and $q(\tau)$ be two given points in σ and let $A(\tau)$, $B(\tau)$ be two open subsets of σ such that*

(a) $\cup_{\tau\in\mathcal{F}} A(\tau) = \cup_{\tau\in\mathcal{F}} B(\tau) = \sigma$.

(b) For each $\tau \in \mathcal{F}$ of dimension $< n$ and for any point $x \in \tau$, there is a $\rho \in \mathcal{F}$ such that $x \in B(\rho)$ and $q(\rho) \in \tau$.

Then there exist two non-empty subfamilies $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ such that

(c) $\left[\cap_{\tau\in\mathcal{G}} A(\tau)\right] \cap \left[\cap_{\rho\in\mathcal{H}} B(\rho)\right] \neq \phi$

and

(d) the convex hull of $\{p(\tau) : \tau \in \mathcal{G}\}$ meets the convex hull of $\{q(\rho) : \rho \in \mathcal{H}\}$.

To prove that Theorems 1 and 2 are equivalent, we first need the following simple lemma:

Lemma. *Let X be a compact Hausdorff space and $A_1, \ldots, A_n$ be open subsets of X such that $X = \cup_{i=1}^n A_i$. Then there exist closed subsets $B_1, \ldots, B_n$ of X such that (i) $B_i \subset A_i$ for each $i = 1, \cdots, n$ and (ii) $\cup_{i=1}^n B_i = X$.*

Proof. For each $y \in X$, let

$$H_y = \cap\{A_i : y \in A_i\},$$

then H_y is an open set containing y; as X is regular, there exists an open set G_y in X such that $y \in G_y \subset \overline{G}_y \subset H_y$. Now as $\{G_y : y \in X\}$ is an open cover of X which is compact, there exist $y_1, \ldots, y_m$ in X for which $X = \cup_{j=1}^m G_{y_j}$. For each $i = 1, \cdots, n$, let

$$B_i = \cup\{\overline{G}_{y_j} : y_j \in A_i\};$$

then B_i is closed in X and $B_i \subset A_i$. The fact that $\cup_{i=1}^n B_i = X$ follows from the fact that $X = \cup_{j=1}^m G_{y_j} = \cup_{i=1}^n A_i$. This completes the proof.

We now show that Theorem 1 and 2 imply each other:

Proof of Theorem 2 From Theorem 1:

By Lemma above, there exist families $\{C(\tau) : \tau \in \mathcal{F}\}$ and $\{D(\tau) : \tau \in \mathcal{F}\}$ of closed subsets of σ such that

(1) $C(\tau) \subset A(\tau)$ and $D(\tau) \subset B(\tau)$ for each $\tau \in \mathcal{F}$,

(2) $\cup_{\tau\in\mathcal{F}} C(\tau) = \cup_{\tau\in\mathcal{F}} D(\tau) = \sigma$.

Fix any $\tau \in \mathcal{F}$ of dimension $< n$. Let

$$\mathcal{F}_\tau = \{\rho \in \mathcal{F} : B(\rho) \cap \tau \neq \phi \text{ and } q(\rho) \in \tau\};$$

then by hypothesis (b), $\tau \subset \bigcup_{\rho\in\mathcal{F}_\tau} B(\rho)$. By Lemma above again, there exists a family $\{D(\tau,\rho) : \rho \in \mathcal{F}_\tau\}$ of closed subsets of τ (and hence also of σ) such that

(3) $D(\tau,\rho) \subset B(\rho)$ for each $\rho \in \mathcal{F}_\tau$,

(4) $\tau = \bigcup_{\rho\in\mathcal{F}_\tau} D(\tau,\rho)$.

Note then for each $x \in \tau$, there exists $\rho \in \mathcal{F}_\tau$ such that

$$x \in D(\tau,\rho) \text{ and } q(\rho) \in \tau. \qquad (^*)$$

Define

$$\mathcal{D} = \{D(\tau) : \tau \in \mathcal{F}\} \cup \{D(\tau,\rho) : \rho \in \mathcal{F}_\tau,\ \tau \in \mathcal{F} \text{ is of dimension } < n\}.$$

For each $\rho \in \mathcal{F}$, let

$$E(\rho) = \bigcup\{H \in \mathcal{D} : H \subset B(\rho)\};$$

then $E(\rho)$ is a closed subset of σ such that $E(\rho) \subset B(\rho)$. Now that $\bigcup_{\rho\in\mathcal{F}} E(\rho) = \sigma$. Now if $\tau \in \mathcal{F}$ is of dimension $< n$ and if $x \in \tau$, by (*) there exists $\rho \in \mathcal{F}_\tau$ such that $x \in D(\tau,\rho)$ and $q(\rho) \in \tau$; it follows that $x \in E(\rho)$ as $D(\tau,\rho) \subset B(\rho)$. Hence by Theorem 1, there exist non-empty subfamilies $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ such that

$$[\bigcap_{\tau\in\mathcal{G}} C(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}} E(\rho)] \neq \phi$$

and the convex hull of $\{p(\tau) : \tau \in \mathcal{G}\}$ meets the convex hull of $\{q(\rho) : \rho \in \mathcal{H}\}$; but then by (1) and (3), we have

$$[\bigcap_{\tau\in\mathcal{G}} A(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}} B(\rho)] \neq \phi.$$

This completes the proof.

Proof of Theorem 1 form Theorem 2:

For each $n = 1, 2 \ldots$ and for each $\tau \in \mathcal{F}$, define

$$A_n(\tau) = \{x \in \sigma : dist\,(x, A(\tau)) < 1/n\},$$
$$B_n(\tau) = \{x \in \sigma : dist\,(x, B(\tau)) < 1/n\},$$

then by Theorem 2, there exist two non-empty subfamilies $\mathcal{G}_n$ and $\mathcal{H}_n$ of $\mathcal{F}$ such that

(1) $[\bigcap_{\tau\in\mathcal{G}_n} A_n(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}_n} B_n(\rho)] \neq \phi$,

(2) the convex hull of $\{p(\tau) : \tau \in \mathcal{G}_n\}$ meets the convex hull of $\{q(\rho) : \rho \in \mathcal{H}_n\}$.

Since $\mathcal{F}$ is finite, there exist non-empty subfamilies $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{F}$ and a sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that $\mathcal{G}_{n_k} = \mathcal{G}$ and $\mathcal{H}_{n_k} = \mathcal{H}$ for each k, and

(1′) $[\bigcap_{\tau\in\mathcal{G}} A_{n_k}(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}} B_{n_k}(\rho)] \neq \phi$ for all $k = 1,2,\ldots,$

(2′) the convex hull of $\{p(\tau) : \tau \in \mathcal{G}\}$ meets the convex hull of $\{q(\rho) : \rho \in \mathcal{H}\}$.

For each $k = 1,2,\ldots,$ choose any $x_k \in [\bigcap_{\tau\in\mathcal{G}} A_{n_k}(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}} B_{n_k}(\rho)]$. By compactness of σ, there is a subsequence $(x_{k_i})_{i=1}^{\infty}$ of $(x_k)_{k=1}^{\infty}$ such that $x_{k_i} \to x$ for some $x \in \sigma$. But then for each $\tau \in \mathcal{G}$,

$$\begin{aligned} dist\,(x, A(\tau)) &\le dist\,(x, x_{k_i}) + dist\,(x_{k_i}, A(\tau)) \\ &< dist\,(x, x_{k_i}) + 1/n_{k_i} \to 0 \qquad \text{as} \qquad i \to \infty, \end{aligned}$$

so that $x \in A(\tau)$ as $A(\tau)$ is closed; thus $x \in \bigcap_{\tau\in\mathcal{G}} A(\tau)$. Similarly we can show that $x \in \bigcap_{\rho\in\mathcal{H}} B(\rho)$. Therefore $[\bigcap_{\tau\in\mathcal{G}} A(\tau)] \cap [\bigcap_{\rho\in\mathcal{H}} B(\rho)] \neq \phi$. This completes the proof.

As an immediate consequence of Theorems 1 and 2, we have the following:

Corollary. *For each $\tau \in \mathcal{F}$, let $q(\tau)$ be a given point in σ and $B(\tau)$ be an open (respectively, a closed) subset of σ such that*

(i) $\bigcup_{\tau\in\mathcal{F}} B(\tau) = \sigma$,

(ii) for each $\tau \in \mathcal{F}$ of dimension $< n$ and for any point $x \in \tau$, there is $\rho \in \mathcal{F}$ such that $x \in B(\rho)$ and $q(\rho) \in \tau$.

Then for each $y \in \sigma$, there exists a non-empty subfamily $\mathcal{H}$ of $\mathcal{F}$ such that

(1) $\bigcap_{\rho\in\mathcal{H}} B(\rho) \neq \phi$,

(2) the convex hull of $\{q(\rho) : \rho \in \mathcal{H}\}$ contains y.

Proof. Take $A(\sigma) = \sigma$ and $A(\tau) = \phi$ for all $\tau \in \mathcal{F}$ with $\tau \neq \sigma$ and take $p(\rho) = y$ for all $\rho \in \mathcal{F}$. The conclusion follows by applying Theorem 2 (respectively, Theorem 1).

In view of Theorem 2 and Fan's observation, by taking $A(\tau) = \phi$ for $\tau \neq \alpha$ and $A(\sigma) = \sigma$ and $p(\tau) = q(\tau)$ is the barycenter $c(\tau)$ of τ for every $\tau \in \mathcal{F}$, we have the following result which is a dual, and is in fact equivalent to Shapley's result [3]:

Theorem 3. *If $\{B(\tau) : \tau \in \mathcal{F}\}$ is a family of open subsets of σ such that $\tau \subset \bigcup_{\tau\subset\rho\in\mathcal{F}} B(\rho)$ for each $\tau \in \mathcal{F}$, then there exists a non-empty subfamily $\mathcal{D}$ of $\mathcal{F}$ such that*

(1) $\bigcap_{\tau\in\mathcal{D}} B(\tau) \neq \phi$,

(2) the convex hull of the set $\{c(\tau) : \tau \in \mathcal{D}\}$ contains the barycenter of σ.

Moreover, by taking $B(\tau) \neq \phi$ only for 0-dimensional faces τ of σ in Theorem 3, we have the following result which is a dual, and is in fact equivalent to the classical Knaster-Kuratowski-Mazurkiewicz Theorem [2]:

Theorem 4. *If $\sigma = a_0a_1\ldots a_n$ and $B_0, B_1, \ldots, B_n$ are open subsets of σ such that for each subset $\{i_0, i_1, \ldots, i_k\}$ of $\{0,1,\ldots,n\}$, $a_{i_0}a_{i_1}\ldots a_{i_k} \subset \bigcup_{j=0}^{k} B_{i_j}$, then $\bigcap_{i=0}^{n} B_i \neq \phi$.*

References

[1] K. Fan, "Some properties of convex sets related to fixed point theorems", *Math. Ann.* 266 (1984), 519-537.

[2] B. Knaster, C. Kuratowski and S. Mazurkiewicz, "Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe", *Fund. Math.* 14 (1929), 132-137.

[3] L. S. Shapley, "On balanced games without side payments", In: *Mathematical Programming*, pp.261-290, (Eds. T. C. Hu and S. M. Robinson) New York, Academic Press, 1973.

Department of Mathematics, National Central University, Chung Li, Taiwan 32054, Republic of China.

Department of Mathematics, Statistics amd Computing Science Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada.