TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 4, Winter 1991

A NOTE ON SIMPLE EXTENSIONS AND SEMI-COMPACT TOPOLOGIES

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Abstract. We study simple extensions of semi-compact topological spaces. Our main result says that if X is an infinite set then maximal semi-compact topologies on X do not exist.

1. Introduction and Preliminaries

Let S be a subset of a topological space (X,τ). The closure and the interior of S with respect to (X,τ) will be denoted by $cl_\tau\, S$ and $\text{int}_\tau\, S$ respectively. If $S \subset cl_\tau(\text{int}_\tau\, S)$ then S is called semi-open [3] in (X,τ), and if $\text{int}_\tau(cl_\tau\, S) = \phi$ then S is said to be nwd (= nowhere dense) in (X,τ). The set of natural numbers will be denoted by ω.

Let A be a subset of a space (X,τ). The simple extension of (X,τ) by A [4] is the space (X,σ) where $\sigma = \{U \cup (V \cap A) : U, V \in \tau\}$. It is pointed out in [4] that for any subset B of X we have $cl_\sigma\, B = cl_\tau B \cap ((X - A) \cup (A \cap cl_\tau(B \cap A)))$. Consequently, $cl_\tau\, B \cap (X - A) \subset cl_\sigma\, B$ for any $B \subset X$, and $cl_\tau\, B = cl_\sigma\, B$ whenever $B \subset A$. From these observations one can infer immediately the following result.

Lemma 1.1 *Let (X,σ) be the simple extension of (X,τ) by $A \subset X$, and let $N \subset X$ be nwd in (X,σ).*

i) *If A is closed and nwd in (X,τ) then N is nwd in (X,τ).*
ii) *If $N \subset A$ then N is nwd in (X,τ).*

2. Semi-Compact Expansions

A space (X,τ) is called semi-compact [2] if every cover of X by semi-open subsets has a finite subcover. Recall that (X,τ) is said to be semi-irreducible [5] if every disjoint family of nonempty open sets is finite. We will make use of the following characterization of semi-compactness which is due to Dorsett [2].

Theorem 2.1. *A space (X,τ) is semi-compact if and only if (X,τ) is semi-irreducible and every nwd subset of (X,τ) is finite.*

Receive November 30, 1990.

Our next result about the preservation of semi-irreducibility will also be needed in the sequel.

Lemma 2.2 *Let (X,τ) be semi-irreducible and let (X,σ) be the simple extension of (X,τ) by $A \subset X$. If A is either dense in (X,τ) or finite then (X,σ) is semi-irreducibile.*

Proof. Suppose that $\{W_n : n \in \omega\}$ is a disjoint family of nonempty open sets in (X,σ). Let $W_n = U_n \cup (V_n \cap A)$ with $U_n, V_n \in \tau$ for each $n \in \omega$. Since (X,τ) is semi-irreducibile there exists $k \in \omega$ such that U_n is empty whenver $n \geq k$. Hence $\{V_n \cap A : n \geq k\}$ is a disjoint family of nonempty sets which is impossible in the case that A is finite. If A is dense in (X,τ) then $V_n \cap V_m \cap A = \phi$ for $n, m \geq k$, and so $\{V_n : n \geq k\}$ is a disjoint family of nonempty open sets in (X,τ), a contradiction.

By using the facts we have carried together so far we are now able to show that certain simple extensions are semi-compact.

Theorem 2.3. *Let (X,τ) be semi-compact and suppose that $\{x_0\}$ is not open for some $x_0 \in X$. If (X,σ) denotes the simple extension of (X,τ) by $A = X - \{x_0\}$ then (X,σ) is semi-compact.*

Proof. Since A is dense in (X,τ), (X,σ) is semi-irreducibile by Lemma 2.2. Let $N \subset X$ be nwd in (X,σ). By Lemma 1.1 ii) $N \cap A$ is nwd in (X,τ) and hence finite. Consequently, N is finite and (X,σ) is semi-compact by Theorem 2.1.

Theorem 2.4. *Let (X,τ) be semi-compact and suppose that $\{x_0\}$ is closed but not open for some $x_0 \in X$. If (x,σ) denotes the simple extension of (X,τ) by $A = \{x_0\}$ then (X,σ) is semi-compact.*

Proof. (X,σ) is semi-irreducibile by Lemma 2.2. Let $N \subset X$ be nwd in (X,σ). Since A is closed and nwd in (X,τ), N is nwd in (X,τ) by Lemma 1.1 i). Hence N is finite and (X,σ) is semi-compact by Theorem 2.1.

3. Maximal Semi-Compact Spaces

If X is a finite set and if τ denotes the discrete topology on X then (X,τ) is obviously maximal semi-compact. It turns out that this is the only possibility for a space to be maximal semi-compact.

Theorem 3.1. *There are no maximal semi-compact topologies on an infinite set.*

Proof. Suppose that (X,τ) is maximal semi-compact and X is infinite. Then (X,τ) is clearly not the discrete space and so there exists $x_0 \in X$ such that $\{x_0\}$ is not open in (X,τ). Since (X,τ) is maximal semi-compact it follows from Theorem 2.3 that $\{x_0\}$ is closed in (X,τ). But now Theorem 2.4 implies that the simple extension of (X,τ) by

$\{x_0\}$ is semi-compact. This obviously produces a contradiction since $\{x_0\}$ is not open in (X, τ).

Remark 3.2. In [1] several results about maximal semi-compact spaces are presented. These results, based on Lemma 2.1 in [1], seem to be contradictory to our results. It should be noted, however, that Lemma 2.1 in [1] is slightly incorrect and a possible correct version would be: "If R and R' are two properties of a space (X, τ) such that R implies R', then (X, τ) is maximal R if it is R and maximal R'".

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TAMKANG JOURNAL OF MATHEMATICS
Volume 22, Number 4, Winter 1991

ON SOME INTEGRAL OPERATORS FOR CERTAIN FAMILIES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, some classes of analytic functions related with functions of bounded boundary rotation are defined and discussed with reference to certain integral operators.

Keywords and Phrases: Subordinate, bounded boundary rotation, starlike, convex, close-to-convex, bounded radius rotation.

1. Introduction

Let f be analytic in $E = \{z : |z| < 1\}$, and given by

$$f(z) \; = \; z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

A function g, analytic in E, is called subordinate to a function G if there exists a Schwarz function $w(z)$ analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E such that $g(z) = G(w(z))$.

In [1], Janowski introduced the class $P[A, B]$. For A and B, $-1 \le B < A \le 1$, a function p, analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. When $A = 1$, $B = -1$, $P[1, -1] \equiv P$, the class of analytic functions with positive real part. We generalize this concept to define the class $P_k[A, B]$, $k \ge 2$. A function $p \in P_k[A, B]$ if, and only if, there exist p_1, $p_2 \in P[A, B]$ such that

$$p(z) \; = \; (\frac{k}{4} + \frac{1}{2}) p_1(z) - (\frac{k}{4} - \frac{1}{2}) p_2(z) \tag{1.2}$$

Definition 1.1. A function f, analytic in E, and given by (1.1) is said to belong to the class $R_k[A, B]$, $k \ge 2$, if and only if, $\frac{zf'(z)}{f(z)} \in P_k[A, B]$.

Receive January 7, 1991.
1980 AMS(MOS) Suject Classfication: 30A32, 30A34, 30C45

For $A = 1$, $B = -1$, we obtain $R_k[1,-1] \equiv R_k$, the class of bounded radius rotation discussed in [2], and $R_2[1,-1] \equiv S^*$, the class of starlike univalent functions. Also $R_2[A,B] \equiv S^*[A,B] \subset S^*$, see [3].

Similarly we have:

Definition 1.2. Let f be analytic in E and given by (1.1). Then $f \in V_k[A,B]$, $k \geq 2$, if and only if $\frac{(zf'(z))}{f'(z)} \in P_k[A,B]$.

If $A = A$, $B = -1$, then $V_k[1,-1] \equiv C$, the class of convex univalent functions.

Also $V_2[A,B] \equiv C[A,B] \subset C$, see [3]. It is clear that

$$f \in V_k[A,B] \Longleftrightarrow zf' \in R_k[A,B] \tag{1.3}$$

Definition 1.3. Let f be analytic in E and given by (1.1). Then f is said to belong to the class $T_k[A,B]$, $k \geq 2$, if and only if, there exists a function $g \in V_k[A,B]$ such that $\frac{f'(z)}{g'(z)} \in P[A,B]$.

We note that:

i. $T_2[1,-1] \equiv K$, the class of close-to-convex functions introduced and studied by Kaplan [4].

ii. $T_k[1,-1] \equiv T_k$, a class of analytic functions introduced and studied in [5].

iii. $T_2[A,B] \equiv K[A,B]$, which has been studied in a more general way in [3].

2. Preliminary Results

Lemma 2.1. *Let $p \in P_k[A,B]$. Then $p \in P$ for $|z| < r_1$, where*

$$r_1 = 4/[k(A-B) + \sqrt{k^2(B-A)^2 + 16AB}]. \tag{2.1}$$

This result is sharp.

Proof. Now $p \in P_k[A,B]$ implies that

$$p(z) = (\frac{k}{4} + \frac{1}{2})p_1(z) - (\frac{k}{4} - \frac{1}{2})p_2(z), \quad p_1, p_2 \in P[A,B].$$

Since

$$\frac{1-Ar}{1-Br} \leq Re p_i(z) \leq | P_i(z) | \leq \frac{1+Ar}{1+Br}, \quad i = 1, 2, \text{ see [6]},$$

we have

$$\begin{aligned} Re\, p(z) &\geq (\frac{k}{4} + \frac{1}{2})\frac{1-Ar}{1-Br} - (\frac{k}{4} - \frac{1}{2})\frac{1+Ar}{1+Br} \\ &= \frac{1 + \frac{k}{2}(B-A)r - ABr^2}{1 - B^2r^2}. \end{aligned}$$

From the above inequality, it is clear that $Re\, p(z) \geq 0$ for $|z| < r_1$, where r_1 is given by (2.1).

The sharpness follows from the function p_0, where

$$p_0(z) = \frac{1}{2}[(\frac{k}{2}+1)\frac{1-Az}{1-Bz} - (\frac{k}{2}-1)\frac{1+Az}{1+Br}].$$

We shall need the following extension of Libera's Lemma [7].

Lemma 2.2. [6]: *Let N and D be analytic in E, D map E noto a manysheeted starlike region. $N(0) = 0 = D(0)$, $N'(0) = 1 = D'(0)$, and*

$$\frac{N'(z)}{D'} \in P[A,B].$$

Then

$$\frac{N(z)}{D(z)} \in P[A,B].$$

Lemma 2.3. [8]: *Let p_1 and $p_2 \in P[A,B]$. Then for α, β any positive reals*

$$\frac{1}{\alpha+\beta}[\alpha p_1(z) + \beta p_2(z)] \in p[A,B].$$

3. Main Results

Theorem 3.1. *Let $f \in V_k[A,B]$ and $g \in R_k[A,B]$. Let H be defined by*

$$H(z) = \int_0^z (f'(t))^\alpha (\frac{g(t)}{t})^\beta dt, \tag{3.1}$$

where α and β are positive reals with $\alpha + \beta = 1$. Then $H \in V_k[A,B]$.

Proof. From (3.1), we have

$$H'(z) = (f'(z))^\alpha (\frac{g(z)}{z})^\beta.$$

Logarithemic differentiation yields

$$\begin{aligned}\frac{(zH'(z))'}{H'(z)} &= \alpha\frac{(zf'(z))'}{f'(z)} + \beta\frac{zg'(z)}{g(z)}\\ &= \alpha p_1(z) + \beta p_2(z),\ p_1, p_2 \in P_k[A,B]\\ &= \alpha[(\frac{k}{4}+\frac{1}{2})q_1(z) - (\frac{k}{4}-\frac{1}{2})q_2(z)] + \beta[(\frac{k}{4}+\frac{1}{2})h_1(z) - (\frac{k}{4}-\frac{1}{2})h_2(z)],\end{aligned}$$

where q_i, $h_i \in P[A,B]$, $i = 1,2$.

Thus, using Lemma 2.3, we have

$$\frac{(zH'(z))'}{H'(z)} = (\frac{k}{4}+\frac{1}{2})q(z) - (\frac{k}{4}-\frac{1}{2})h(z),$$

q, $h \in P[A,B]$ and this gives us the required result that $H \in V_k[A,B]$.

Following the same technique used in Theorem 3.1, we have:

Theorem 3.2. *Let $f_i \in V_k[A,B]$, $i = 1,2,\ldots,n$. Then*

$$K(z) = \int_0^z \pi_i(f_1'(t))^{\alpha_i} dt, \qquad \sum_{i=1}^{n} \alpha_i = 1$$

belongs to $V_k[A,B]$.

Theorem 3.3. *Let $f \in R_k[A.B]$ and let*

$$F(z) = [(\alpha+\nu+\eta)z^{-\nu} \int_0^z (f(t))^{\alpha} t^{\nu+\eta-1} dt]^{\frac{1}{\alpha+\eta}}, \tag{3.2}$$

where $\alpha > 0$, $\eta \geq 0$ and $Re(\nu+\eta) \geq 0$. Then F maps $|z| < r_1$ onto a star-shaped domain, r_1 is given by (2.1).

In (3.2) all powers are principal ones.

Proof. Differentiating (3.2) logarithemically, we obtain

$$\frac{zF'(z)}{F(z)} = \frac{f^{\alpha}(z)z^{\nu+\eta} - \nu\int_0^z f^{\alpha}(t)t^{\nu+\eta-1}dt}{(\alpha+\eta)\int_0^z f^{\alpha}(t)t^{\nu+\eta-1}dt} = \frac{N(z)}{D(z)}.$$

We note that $N(0) = 0 = D(0)$, and by a Lemma due to Bernardi [9], $D(z)$ is $(\nu+\eta+\alpha-1)$ - valent starlike for $|z| < r_1$, where r_1 is given by (2.1). Also

$$\frac{N'(z)}{D'(z)} = \frac{\alpha z f'(z)}{(\alpha+\eta)f(z)} + \frac{\eta}{\alpha+\eta},$$

that is

$$\frac{N'(z)}{D'(z)} \in P[1-\frac{2\eta}{\alpha+\eta},-1]$$

for all $|z| < r$. Using lemma 2.2 with $A = 1-\frac{2\eta}{\alpha+\eta}$, $B = -1$, we see that $\frac{N(z)}{D(z)} \in P[1-\frac{2\eta}{\alpha+\eta} - 1]$ for $|z| < r_1$. Hence $F \in S^*[1-\frac{2\eta}{\alpha+\eta},-1] \subset S^*$ for $|z| < r$, and this gives us the required reult.

From the relationship (1.3) and the fact that f is convex if and only if zf' is starlike, we immediately have the following:

Theorem 3.4. *Let $f \in V_k[A,B]$. Then F, defined by (3.2), maps $|z| < r_1$ onto a convex domain, where r_1 is given by (2.1).*

Theorem 3.5. *Let $f \in T_k[A,B]$ with respect to $h \in V_k[A,B]$. Let $g \in R_k[A,B]$, and for $\alpha + \beta = 1$, $\alpha, \beta \geq 0$, let F be defined as*

$$F(z) = \int_0^z (f'(t))^\alpha (\frac{g(t)}{t})^\beta dt.$$

Then F is close-to-convex with respect to H defined by

$$H(z) = \int_0^z (h'(t))^\alpha (\frac{g(t)}{t})^\beta dt$$

for all $|z| < r$, r_1 given by (2.1).

Proof. From Theorem 3.1, we see that $H \in V_k[A,B]$.
Now

$$\begin{aligned} \frac{F'(z)}{H'(z)} &= (f'(z))^\alpha (\frac{g(z)}{t})^\beta / (h'(z))^\alpha (\frac{g(z)}{z})^\beta \\ &= (\frac{f'(z)}{h'(z)})^\alpha = p_1^\alpha(z) \in P[A,B],\ 0 \leq \alpha \leq 1. \end{aligned}$$

Hence the results.

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