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A NOTE ON FLAT MODULES OVER f-ALGEBRAS

BORIS LAVRIČ

Abstract. Let A be an Archimedean uniformly complete unital f-algebra. It is proved that the following conditions are equivalent: (1) A is a Bezout ring; (2) A is a PF-ring; (3) Every ideal of A is flat; (4) Every submodule of a free A-module is flat. This extends a result by C. Neville on algebras of type $C(X)$.

Introduction

Let X be completely regular Hausdorff topological space, and let $C(X)$ be the ring of all continuous real-valued functions on X. Using the well known topological characterization of spaces X for which $C(X)$ is a Bezout ring [5] C. Neville has proved in [7] that the following conditions are equivalent.

(1) $C(X)$ is a Bezout ring.

(2) $C(X)$ is a PF-ring.

(3) Every ideal of $C(X)$ is flat.

(4) Every submodule of a free $C(X)$-module is flat.

It is the aim of this note to extend this result to uniformly complete Archimedean unital f-algebras.

For the terminology and general theory of rings and modules we refer the reader to [1], [4], and for elementary f-algebra theory we refer to [6] and [8].

All rings considered in the paper are assumed to be commutative and with unit element. A ring R is called a *Bezout ring* whenever every finitely generated ideal of R is principal. A ring R is called a *PF-ring* if every principal ideal of R is a flat R-module. A ring R is said to be *reduced* (or *semiprime*) if it has no nonzero nilpotent elements.

A lattice ordered real algebra A is called an *f-algebra* whenever

$$a \wedge b = 0,\ a, b \in A \text{ implies } ac \wedge b = ca \wedge b = 0$$

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for all $c \in A^+$. It is well known that an Archimedean f-algebra A with unit is commutative, reduced, and that $a, b \in A$ satisfy $ab = 0$ if and only if $|a| \wedge |b| = 0$. An f-algebra A is said to be *normal* if

$$a, b \in A^+, \ a \wedge b = 0 \text{ implies } \{a\}^d + \{b\}^d = A,$$

where $G^d = \{h \in A : |g| \wedge |h| = 0 \text{ forall } g \in G\}$ for $G \subset A$. It follows that an Archimedean unital f-algebra A is normal if and only if

$$ab = 0, \ a, b \in A \text{ implies } ann(a) + ann(b) = A.$$

Results

First we state a criterion characterizing those principal ideals of a ring R which are flat R-modules. Since the criterion is an easy consequence of [2, Prop. 2.3] or [1, I.2.11. Cor.1], we omit its proof.

Lemma 1. *A principal ideal (a) of a ring R is a flat R-module if and only if for each $b \in R$ satisfying $ab = 0$ there exist elements $a_1, a_2, \dots, a_n \in (a)$ and $b_1, b_2, \dots, b_n \in R$ such that*

$$a = \sum_{i=1}^{n} a_i b_i \text{ and } b_i b = 0, \ i = 1, 2, \cdots, n.$$

Proposition 1. *For a ring R the following conditions are equivalent.*
(i) *R is a PF-ring.*
(ii) *$a, b \in R$, $ab = 0$ implies that $ann(a) + ann(b) = R$.*

Proof. (i)$\Longrightarrow$(ii). Let $a, b \in R$ satisfy $ab = 0$. If R is a PF-ring, then (a) is flat, hence by Lemma 1 we have

$$a = \sum_{i=1}^{n} a_i b_i \text{ and } b_i b = 0, \ i = 1, 2, \cdots, n$$

for some $a_1, \dots, a_n \in (a)$, $b_1, \dots, b_n \in R$. Write

$$a_i = c_i a, \ c_i \in R, \ i = 1, 2 \cdots, n.$$

It follows that $a = (\Sigma_{i=1}^{n} b_i c_i) a$, and therefore

$$1 - \sum_{i=1}^{n} b_i c_i \in ann(a), \ \sum_{i=1}^{n} b_i c_i \in ann(b),$$

so (ii) follows.

(ii)$\Longrightarrow$(i). We shall prove that (a) is flat for each $a \in$R. Suppose that $b \in R$ satisfies $ab = 0$. By (ii) there exist $c \in ann(a)$ and $d \in ann(b)$ with $c + d = 1$. So

$$a = a1 = ac + ad = ad \text{ and } db = 0,$$

thus by Lemma 1 (a) is flat.

Corollary. *Let A be an Archimedean unital f-algebra. Then A is a PF-ring if and only if A is normal.*

Since by [6] $C(X)$ is normal if and only if X is an F-space, the above corollary generalizes [7, Cor.1.7].

Lemma 2. *Let R be a reduced Bezout ring. Then every ideal of R is flat.*

Proof. Since a module is flat if and only if every finitely generated submodule is flat, and since R is Bezout, it suffices to show that R is a PF-ring.

Now for any $a, b \in R$ with $ab = 0$, we shall check that $ann(a) + ann(b) = R$.

Since $(a, b) = (c)$ for some $c \in R$, we have

$$a = a_1c, \ b = b_1c, \ c = a_2a + b_2b$$

for some $a_1, b_1, a_2, b_2 \in R$. Since R is reduced,

$$(a_1b_1c)^2 = a_1b_1(ab) = 0$$

implies $a_1b_1c = 0$. It follows that $d = a_1a_2$ satisfies

$$bd = b_1ca_1a_2 = 0$$

and

$$\begin{aligned} a(1-d) &= a - aa_1a_2 = a_1c - aa_1a_2 \\ &= a_1(a_2a + b_2b) - aa_1a_2 = a_1b_1cb_2 = 0. \end{aligned}$$

Hence

$$1 = (1-d) + d \in ann(a) + ann(b),$$

and therefore $ann(a) + ann(b) = R$.

We are now in a position to extend a part of theorem 3.1 from [7] on modules over reduced Bezout rings. A short proof suggested by the referee is based on [3]. For the sake of completeness we shall repeat briefly the arguments used in the proof of [3, V.Lemma 6.8].

Proposition 2. *Let R be a reduced Bezout ring. Then every submodule of a free R-module is flat.*

Proof. Let F be a free module with base X, and let M be its submodule. Assuming the axiom of choice we may suppose the elements of X are well ordered: $x_1, x_2, \ldots, x_\alpha, \ldots$. For each ordinal α, let M_α be the submodule of those elements of M which are linear combinations of the x_β with $\beta \leq \alpha$, and let N_α be the submodule of all linear combinations of the x_β with $\beta < \alpha$.

Each $m \in M_\alpha$ is decomposed uniquely into

$$m = n + r(m)x_\alpha, \; n \in N_\alpha, \; r(m) \in R,$$

so M_α is a direct sum of N_α and $K_\alpha = r(M_\alpha)x_\alpha$. Observe that $I_\alpha = r(M_\alpha)$ is an ideal of R, and that M is a direct sum of all K_α [3]. Since by Lemma 2 each K_α is flat, it follows by [1, I.2. Prop.2] that M is flat.

We are prepared to prove the main result of the present note.

Theorem. *Let A be an Archimedean uniformly complete unital f-algebra. Then the following conditions are equivalent.*

(i) *A is a Bezout ring.*
(ii) *A is a PF-ring.*
(iii) *Every ideal of A is flat.*
(iv) *Every submodule of a free A-module is flat.*

Proof. The implication (i)$\Longrightarrow$(iv) follows by Proposition 2, while (iv)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii) are obvious. To prove (ii)$\Longrightarrow$(i) suppose that A is a PF-ring. Then by Corollary A is normal, hence Bezout by [6, Theorem 6.6].

Some other characterizations of an f-algebra satisfying the conditions of the above theorem are contained in [6], where also the following example is given. It shows that the uniform completeness cannot be dropped from the Theorem.

Example. Let A be the f-algebra of all real functions f on the aquare $E = [0,1] \times [0,1]$ for which there exist disjoint sets $E_1, \ldots, E_{n(f)}$ such that $E = E_1 \cup \ldots E_{n(f)}$, and polynomials $p_k \in \mathbb{R}[X,Y]$ satisfing

$$f\,|_{E_k} = p_k, \; k+1, \ldots, n(f).$$

Clearly A is an Archimedean unital f-algebra. By [6] A is normal, hence a PF-ring, although A is not a Bezout ring [6]. We claim that A does not satisfy the condition (iii) of the Theorem.

Let I be the ideal generated by elements

$$f, g \in A, \; f(x,y) = x, \; g(x,y) = y.$$

We will show that I is not flat. Consider the exact sequence

$$0 \longrightarrow K \longrightarrow A \oplus A \xrightarrow{\varphi} I \longrightarrow 0,$$

where φ is defined by $\varphi(a,b) = af + bg$, and suppose that I is flat. Then by [2, Proposition 2.2] or by [4, 11.27] there exists a homomorphism $\psi : A \oplus A \longrightarrow K$ such that $\psi(g,-f) = (g,-f)$. If

$$\psi(1,0) = (a_1 b_1), \ \psi(0,1) = (a_2, b_2),$$

then

$$a_1 f + b_1 g = a_2 f + b_2 g = 0,$$

$$g = a_1 g - a_2 f, \ -f = b_1 g - b_2 f.$$

Using the fact that these equalities cannot hold in $\mathbb{R}[X,Y]$ it is not difficult to see that they are contradictive also in A, so I is not flat.

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Institute of Mathematics, Physics and Mechanics, Jadranska 19, 61000 Ljubljana, Yugoslavia.