TAMKANG JOURNAL OF MATHEMATICS
Volume 21, Number 1, Spring 1990

LOCAL DISCRETE EXTENSIONS OF SUPRATOPOLOGIES

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Abstract. In this paper, we introduce the concept of local discrete extensions of supratopologies on a set. The basic problem is to investigate the supratopological properties that are preserved under local discrete extensions.

1. Introduction

Let (X, τ) be a topological space on which no separation axioms are stated and whenever such axioms are needed they will be explicitly assumed. A class $\tau^* \subset P(X)$ is called a *supratopology* [2] on X if $X \in \tau^*$ and τ^* is closed under arbitrary union. (X, τ^*) is called *supratopological space* (briefly a *supraspace*). A supratopology τ^* is called *associated with* τ if $\tau \subset \tau^*$ and each member of τ^* is called a *supraopen set* and the complement of a suparopen set is called *supraclosed* [2]. Various notions like interior, closure, exterior and the derived set operators can be defined in supratopological spaces in analogy with topological spaces [2]. The supraderived set (resp. supraclosure, suprainterior) of a subset A of a space X will be denoted by $d_{\tau^*}A$ (resp. $cl_{\tau^*}A$, $int_{\tau^*}A$). Also, Mashhour, et. al [2] have introduced the concept of $S-T_i (i = 0, 1, 2)$ and $S-T_2'$ separation axioms in supraspaces, by replacing open sets by supraopen sets in T_i and T_2' separation axioms. By the same manner they introduced the concept of S^*-regularity and S^*-normality [3]. In [2] the concept of S^*-continuity was defined as follows : A function $f : (X, \tau_1^*) \rightarrow (Y, \tau_2^*)$ is $S^* -$ *continuous* if the inverse image of each τ_2^*-supraopen set is τ_1^* supraopen.

In 1971, Young, S. P. [4] introduced the concept of local discrete extensions of topologies. Let (X, τ) be a topological space and A be a subset of X. Then the topology $\tau[A] = \{U - B : U \in \tau, B \subset A\}$ is called *local discrete extension* of τ by A. He attampted to investigate, if (X, τ) has some topological property Q, under what conditions will $(X, \tau[A])$ also have property Q.

The purpose of the present paper is to introduce the concept of local discrete extensions of supratopologies on a set and to study the preservation of some supratopological properties under local discrete extensions, in a way analogous to results obtained by Young [4]. Also, we introduce and study the concept of the base for supratopologies, the local base at a point in a supraspaces, the first countable and the second contable

Received September 21, 1987; revised Appil 11, 1989.

supraspaces and study the preservation of such supraspaces under local discrete extensions.

2. Local discrete extensions of supratopologies

Definition 2.1. Let (X,τ^*) be supraspace and $A \subset X$, then $\tau^*[A]$ is called *local discrete extension* of τ^* by A iff $\tau^*[A] = \{U - B : U \in \tau^*,\ B \subset A\}$.

It is clear that $\tau^*[A]$ is a supratopology on X and $\tau^* \subset \tau^*[A]$.

Remark 2.1. (i) If τ is topology on X and τ^* is an associated supratopology with τ, then $\tau[A] \subset \tau^*[A]$, where $\tau[A]$ is the local discrete extension of τ by A in the sense of Young [4]. The inclusion relation cannot be replaced by equality sign, in general, as shown by Example 2.1.

(ii) *The concept of local discrete extensions and simple extensions of supratopologies [1] are independent concepts (Example 2.2).*

Example 2.1. Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\}\}$ and supratopology $\tau^* = \{X,\phi,\{a\},\{a,c\},\{b,c\},\{a,b,c\}\}$, For $A = \{b,d\}$, $\tau[A] = \{X,\phi,\{a\},\{a,c\},\{a,b,c\},\{a,c,d\}\}$ and $\tau^*(A) = \{X,\phi,\{a\},\{a,c\},\{a,b,c\},\{a,c,d\},\{c\},\{b,c\}\}$. Therefore, $\tau[A] \neq \tau^*[A]$.

Example 2.2. Let $X = \{a,b,c\}$ with topology $\tau = \{X,\phi,\{a\},\{a,c\}\}$ and supratopology $\tau^* = \{X,\phi,\{a\},\{a,c\},\{b,c\}\}$. For $A = \{a,b\}$, $\tau^*[A] = \{X,\phi,\{a\},\{a,c\},\{b,c\},\{c\}\}$ and $\tau^*(A) = \{X,\phi,\{a\},\{a,c\},\{b,c\},\{a,b\},\{b\}\}$.

Proposition 2.1. *If (X,τ^*) is a supraspace and $A \subset X$, then $(A,\tau^*[A] \cap A)$ is discrete.*

Proof. To prove that $(A,\tau^*[A] \cap A)$ is discrete for any $A \subset X$, we need to show that every singletion $\{p\} \subset A$ is both open and closed in $(A,\tau^*[A] \cap A)$. Let $\{p\} \subset A$. Then $X - \{p\} \in \tau^*[A]$ and $(X - \{p\}) \cap A = A - \{p\} \in \tau^*[A] \cap A$ and hence $\{p\}$ is supraclosed in A. On the other hand, $A - \{p\} \subset A$ implies $X - (A - \{p\}) \in \tau^*[A]$ and $X - (A - \{p\}) \cap A = A - (A - \{p\}) = \{p\} \in \tau^*[A] \cap A$.

Proposition 2.2. *For a supraspace (X,τ^*), $\tau^*[A] \supset \tau^*[B]$ for any $B \subset A$.*

Proof. $\tau^*[B] = \{U - C : U \in \tau^*,\ C \subset B \subset A\} \subset \tau^*[A]$. The inclusion relation in Proposition 2.2 cannot be replaced by equality sign as shown by the following example.

Example 2.3. Let $X = \{a,b,c,d\}$ with topology $\tau = \{X,\phi,\{a\},\{a,d\}\}$ and supratopology $\tau^* = \{X,\phi,\{a\},\{a,d\},\{a,c\},\{a,c,d\}\}$. For $A = \{a,b,d\}$ and $B = \{b\} \subset A$, $\tau^*[A] = \{X,\phi,\{a\},\{a,d\},\{a,c\},\{a,c,d\},\{c\},\{d\},\{c,d\}\}$ and $\tau^*[B] = \{X,\phi,\{a\},\{a,d\},\{a,c\},\{a,c,d\}\}$.

Theorem 2.1. *If (X,τ^*) is a supraspace, $\tau^*[A]$ is a local discrete extension of τ^* and B is any subset of X, then*

(i) $cl_{\tau^*[A]}B = (A \cap B) \cup cl_{\tau^*}((X - A) \cap B)$.
(ii) $int_{\tau^*[A]}B = ((X - A) \cap B) \cap int_{\tau^*}(A \cup B)$.
(iii) $d_{\tau^*[A]}B \subset d_{\tau^*}B$ ($d_{\tau^*[A]}$, d_{τ^*} *denote the derived operator relative to* $\tau^*[A]$ *and* τ^*, *respectively*).

Proof. Proofs of (i) and (ii) follows in a similar manner to the topological case considered by Young [4]. (iii) follows from $\tau^* \subset \tau^*[A]$.

The inclusion relation in ((iii) Theorem 2.1) cannot be replaced by equality sign as illustrated by the following example.

Example 2.4. In Example 2.3, consider $B = \{a, c\}$, then $d_{\tau^*[A]}B = \{b\}$ and $d_{\tau^*}B = \{b, c, d\}$. Hence $d_{\tau^*[A]}B \not\supset d_{\tau^*}B$.

3. Preservation of some supratopological properties under local discrete extensions of supratopologies

In what follows we discuss the preservation of some supratopological properties, $S - T_i (i = 0, 1, 2)$, $S - T_2'$ axioms, S^*-regularity and S^*-normality, under local discrete extensions of supratopologies.

Theorem 3.1. *If* (X, τ^*) *is* $S - T_2$ *(resp.* $S - T_1$, $S - T_0$*) supraspace, then* $(X, \tau^*[A])$ *is* $S - T_2$ *(resp.* $S - T_1, S - T_0$*).*

Proof. Obvious, since $\tau^* \subset \tau^*[A]$.

The converse of Theorem 3.1 is false, in general, as shown by the following example.

Example 3.1. Let $X = \{a, b, c\}$ with topology $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and supratopology $\tau^* = \{X, \phi, \{a\}, \{b, c\}, \{a, c\}\}$. For $A = \{c\}$, $\tau^*[A] = \{X, \phi, \{a\}, \{b, c\}, \{a, c\}, \{a, b\}, \{b\}\}$. $(X, \tau^*[A])$ is an $S - T_2$ supraspace while (X, τ^*) is not.

Theorem 3.2. *If* (X, τ^*) *is* $S - T_2'$, *then* $(X, \tau^*[A])$ *is* $S - T_2'$.

Proof. Let (X, τ^*) be an $S - T_2'$ supraspace and x, y be two distinct points of X. Then there exist two supraopen sets $U, V \in \tau^* \subset \tau^*[A]$ containing x and y, respectively, such that $cl_{\tau^*}U \cap cl_{\tau^*}V = \emptyset$. Hence, $cl_{\tau^*[A]}U \cap cl_{\tau^*[A]}V \subset cl_{\tau^*}U \cap cl\ V_{\tau^*} = \emptyset$ and $(X, \tau^*[A])$ is an $S - T_2'$ supraspace.

Theorem 3.3. *If* (X, τ^*) *is* S^**-regular (*S^**-normal) and* A *is a supraopen subset of* X, *then* $(X, \tau^*[A])$ *is* S^**-regular (*S^**-normal).*

Proof. Let A be a supraopen subset of (X, τ^*) then every subset A is $\tau^*[A]$ supraopen set. Let F be a supraclosed subset of $(X, \tau^*[A])$ and $x \notin F$. Then there exists a $\tau^*[A]$ supraopen set $U - B$, $U \in \tau^*$, $B \subset A$, such that $F = X - (U - B) = (X - U) \cup (X \cap B) = (X - U) \cup B$, $x \notin F$. Hence $x \notin X - U$ and $x \notin B$. There are two cases (i) $x \notin A$. Since (X, τ^*) is S^*-regular, for each $x \notin X - U$, there exist disjoint supraopen sets U and V such that $x \in U$ and $X - U \subset V$. Hence there are disjoint $\tau^*[A]$ supraopen sets $U - A$ and $V \cap U$ such that $x \in U - A$ and $F \subset V \cap B$. (ii) $x \in A$. Since (X, τ^*) is S^*-regular, there exist disjoint τ^*-supraopen sets, consequently $\tau^*[A]$ supraopen, A and V such that $x \in A$ and $X - A \subset V$. Therefore, $(X, \tau^*[A])$ is S^*-regular.

In case that A is not supraopen subset of (X, τ^*), the above theorem does not hold in general.

Example 3.2. Let $X = \{a, b, c\}$ with indiscrete supratopology $\tau^* = \{X, \phi\}$. Then for $A = \{a, b\}$, $\tau^*[A] = \{X, \phi, \{c\}, \{a, c\}, \{b, c\}\}$ and hence (X, τ^*) is S^*-regular and S^*-normal while $(X, \tau^*[A])$ is neither.

4. Bases for supratopologies

Definition 4.1. Let (X, τ^*) be a supraspace. A class β^* of supraopen sets of X, i.e. $\beta^* \subset \tau^*$ is *a base for the supratopology* iff every supraopen set $G \in \tau^*$ is the union of numbers of β^*, equivalently, for any point $p \in G$, $G \in \tau^*$, there exists a member $B_p \in \beta^*$ such that $p \in B_p \subset G$. Clearly in any supraspace (X, τ^*), τ^* is a base for itself.

Theorem 4.1. *Let β^* be a class of subsets of a nonempty set X. Then β^* is a base for some supratopology τ^* on X iff $X = \cup\{B : B \in \beta^*\}$.*

Proof. Suppose β^* is a base for a supratopology τ^* on X. Since X is supraopen, X is the union of members of β^*, i.e. $X = \cup\{B : B \in \beta^*\}$. Conversely, let β^* be a class of subsets of X satisfy $X = \cup\{B : B \in \beta^*\}$. Let τ^* be the class of all subsets of X which are unions of members of β^*. We have prove that τ^* is a supratopology an X. Since $X = \cup\{B : B \in \beta^*\}$, $X \in \tau^*$. Also, ϕ is the union of an empty subclass of β^*, i.e. $\phi = \cup\{B : B \in \phi \subset \beta^*\}$, hence $\phi \in \tau^*$. Moreover let $\{G_i\}$ be a class of members of τ^*. By definition of τ^*, each $\cup_i G_i$ is the union of members of β^*, hence the union $\cup_i G_i$ is also a union of members of β^*. So, $\cup_i G_i \in \tau^*$. Therefore, τ^* is a supratopology on X.

Theorem 4.2. *If β is a base for a supratopology τ^* on X and β^* is a class of supraopen sets containing β, i.e. $\beta \subset \beta^*$. Then β^* is a base for τ^*.*

Proof. Let G be a supraopen subset of X. Since β is a base for τ^*, G is the union of members of β, i.e. $G = \cup_i B_i$ where $B_i \in \beta$. But $\beta \subset \beta^*$, hence each $B_i \in \beta$, is also belongs to β^*. So G is the union of members of β^* and therefore, β^* is also a base for τ^*.

Definition 4.2. A class β_p^* of supraopen subsets of a supraspace (X, τ^*) containing $p \in X$ is called a *local base at* p iff for each supraopen set G containing p, there exists $G_p \in \beta_p^*$ such that $p \in G_p \subset G$.

Theorem 4.3. *A point $p \in X$ belongs to the supraderived set of $A \subset X (p \in d_{\tau^*} A)$ iff each member of some local base β_p^* at p contains some points of X deifferent from p.*

Proof. Let $p \in d_{\tau^*} A$, then $(G - \{p\}) \cap A \neq \emptyset$ for all $G \in \tau^*$, $p \in G$. But $\beta_p^* \subset \tau^*$, so inparticular $(B - \{p\}) \cap A \neq \emptyset$ for all $B \in \beta_p^*$. Conversely, suppose $(B - \{p\}) \cap A \neq \emptyset$ for all $B \in \beta_p^*$ and let G be any supraopen subset of X containing p. Then there exists $B_0 \in \beta_p^*$ for which $p \in B_0 \subset G$. Then $(G - \{p\}) \cap A \supset (B_0 - \{p\}) \cap A \neq \emptyset$. So, $(G - \{p\}) \cap A \neq \emptyset$ and hence $p \in d_{\tau^*} A$.

Definition 4.3. Let (X, τ^*) be a supraspace. A sequence $< a_n >$, $n \in N$ S^*- *converges* to a point $b \in X$ iff for each supraopen set G containing b, there exists $n_0 \in N$ such that for all $n \geq n_0$ implies $a_n \in G$, i.e. if $G \in \tau^*$ contains almost all, i.e. all except finite member of the terms of the sequence.

Theorem 4.4. *A sequence $< a_n >$ of points in a supraspace (X, τ^*) S^*-converges to $p \in X$ iff each members of some local base β_p^* at p contains almost all the terms of the sequence.*

Proof. $< a_n >$ S^*-converges to $p \in X$ iff every supraopen set G containing P contains almost all the terms of the sequence. But $\beta_p^* \subset \tau^*$, so inparticular each $B \in \beta_p^*$ contains almost all the terms of the sequence. Conversely, suppose every $B \in \beta_p^*$ contains almost all the terms of the sequence and let G be any supraopen set containing p. Then there exists $B_0 \in \beta_P^*$ for which $p \in B_0 \subset G$. Hence G is also contains almost all the terms of the sequence. Therefore, $< a_n >$ S^*-converges to p.

5. Preservation of first countable and second countable supraspaces under local discrete extensions of supratopologies

Definition 5.1. A supraspace (X, τ^*) is *a first countable supraspace* iff there exists a countable local base at every $x \in X$.

Definition 5.2. A supraspace (X, τ^*) is *a second countable supraspace* iff there exists a countable base β^* for the supratopology.

Theorem 5.1. *If $f : (X, \tau^*) \overset{onto}{\longrightarrow} (Y, U^*)$ is an S^*-continuous mapping and (X, τ^*) is a second (first) countable supraspace, then (Y, U^*) is a second (first) countable supraspace.*

Proof. We prove the theorem only for the second countable supraspace.

Let (X, τ^*) be a second countable supraspace and let G be a supraopen subset in (Y, U^*). Then $f^{-1}(G) \in \tau^*$, because f is S^*-continuous, and hence $f^{-1}(G) = \cup\{H_\lambda : \lambda \in \Lambda \subset N\}$ where N is a countable index set and the family $\{H_m, m \in M\}$ is a countable base for τ^*. Since f is onto, $ff^{-1}(G) = G$ and $G = \cup\{f(H_\lambda)\}$, it follows that the countable family of supraopen sets $\{f(H_m), m \in M\}$ is a base for U^*.

Theorem 5.2. *If A is a subset of a second countable supraspace (X, τ^*), then every supraopen cover of A is reducible to a countable cover.*

Proof. Let $\mathcal{U} = \{U_\lambda, \lambda \in \Lambda\}$ be a supraopen cover of A, i.e. $A \subset \cup\{U : U \in \mathcal{U}\}$ and let β be a countable base for X. Hence for $x \in A$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Since β is a base for τ^*, for every $x \in A$, there exists $B_x \in \beta$ such that $x \in B_x \subset U_x$. Hence $A \subset \cup\{B_x : x \in A\}$. But $\{B_x : x \in A\} \subset \beta$, so it is countable, hence $\{B_x : x \in A\} = \{B_n, n \in N\}$ where N is a countable index set. For each $n \in N$ choose one set $U_n \in \mathcal{U}$ such that $B_n \subset U_n$. Then $A \subset \{B_n : n \in N\} \subset \cup\{U_n : n \in N\}$. Therefore, $\{U_n, n \in N\}$ is a countable cover of $\mathcal{U}$.

The following theorems discuss the preservation of the first countable and the second countable supraspace under local discrete extensions of supratopologies.

Theorem 5.3. *If A is any subset of a supraspace (X, τ^*), then (X, τ^*) is a first countable iff $(X, \tau^*[A])$ is a first countable surpraspace.*

Proof. Let $\{U_i, i = 1, 2, \ldots\}$ be a countable local base at any point x of (X, τ^*). There are two cases. (i) $x \notin A$. Then $\{U_i - A, i = 1, 2, \ldots\}$ is a countable local base of a point x of $(X, \tau^*[A])$. (ii) $x \in A$. Then $\{U_i - (A - \{x\}) : i = 1, 2, \ldots\}$ is a countable local base of a point x of $(X, \tau^*[A])$. Hence $(X, \tau^*[A])$ is a first countable supraspace.

Theorem 5.4. *If (X, τ^*) is a second countable supraspace, then $(X, \tau^*[A])$ is a second countable if A is a countable subset of (X, τ^*).*

Proof. Let $\mathcal{U}$ be a countable base for (X, τ^*) and $\mathcal{U}_A = \{B - A_\alpha : B \in \mathcal{U}, A_\alpha$ is a cofinite subset of A$\}$. Then $\mathcal{U}_A$ is countable and a base for $\tau^*[A]$. For, let $x \in B - A_1$, and $A_1 \subset A$. If A_1 is a cofinite subset in A, the proof follows directly. If A_1 is not cofinite subset in A, then $A - A_1$ is not finite and we have two cases (i) $x \notin A$. Then $x \notin B - A \subset B - A_1$. (ii) $x \in A$. Then $x \in B - (A - \{x\}) \subset B - A_1$. Therefore, $(X, \tau^*[A])$ is a second countable supraspace.

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