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ON BOUNDEDLY COMPLETE TRANSFINITE BASES IN BANACH SPACES

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1. Introduction

Initially, Bessaga [1] defined the concept of monotone transfinite basis which, in fact he called '*monotone basis of type* ν'. Subsequently, Bessaga [2] replaced the condition of monotonicity by a weaker condition of uniform boundedness and introduced the term '*projection basis of type* ν'. However, the definition of the transfinite basis, we use in the present note is due to Dorembus [5] who, in fact, worked directly with transfinite bases of subspaces.

In the present note we consider two properties, (I) and (II), in Banach spaces having transfinite bases and observe that they are equivalent. We also prove that if a transfinite basis satisfies (II) (or (I)), then the transfinite basis is boundedly complete in the sense of Bessaga [3] and an example has been exhibited to show that the converse need not be true. This paper may be regarded in a sequel to [6].

2. Preliminaries and lemmas

Throughout E will denote a Banach space over the scalar field $\mathbf{K}$($\mathbf{R}$ or $\mathbf{C}$), ν an infinite ordinal and $[x_\lambda]_{\lambda<\nu}$ the closed linear span of $\{x_\lambda\}_{\lambda<\nu}$.

For the terms like transfinite biorthogonal system, associated transfinite sequence of coefficient functionals (a.t.s.c.f), convergence of a transfinite series, transfinite basis, boundedly complete transfinite basis, sequence of partial sum operators $\{S_\lambda\}_{\lambda<\nu}$ and uniformly convex Banach space, one may refer to [7].

Lemma [6]. *Let $\{x_\lambda\}_{\lambda<\nu}$ be a transfinite sequence of non-zero elements in E such that $[x_\lambda]_{\lambda<\nu}=E$. Then $\{x_\lambda\}_{\lambda<\nu}$ is a transfinite basis in E if and only if there exists a constant $M\geq 1$ such that*

$$\|\sum_{\chi<\lambda}\alpha_\chi x_\chi\| \leq M\|\sum_{\chi<\mu}\alpha_\chi x_\chi\|,$$

for all ordinals λ, μ with $\lambda\leq\mu<\nu$ and arbitrary scalars $\alpha_1,\alpha_2,\ldots,\alpha_\mu$ in $\mathbf{K}$ such that $\sum_{\chi<\mu}\alpha_\chi x_\chi$ converges.

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The infimum of M, denoted by $V_{\{x_\lambda\}_{\lambda<\nu}}$, is called the *norm* of the transfinite basis $\{x_\lambda\}_{\lambda<\nu}$. In case, $V_{\{x_\lambda\}_{\lambda<\nu}} = 1$, the transfinite basis $\{x_\lambda\}_{\lambda<\nu}$ is called *monotone*.

3. Main results

Let $\{x_\lambda\}_{\lambda<\nu}$ be a transfinite basis in E. Then :
(a) $\{x_\lambda\}_{\lambda<\nu}$ is said to *satisfy property* (I) if for each $c > 0$, there exists a number $\gamma_c > 0$ (independent of $\{\alpha_\lambda\}_{\lambda<\nu}$ and μ) such that

$$\|\sum_{\chi\leq\mu} \alpha_\chi x_\chi\| = 1,\ \|\sum_{\mu+1\leq\chi<\nu} \alpha_\chi x_\chi\| \geq c \Rightarrow \|\sum_{\chi<\nu} \alpha_\chi x_\chi\| \geq 1+\gamma_c,$$

where $\{\alpha_\lambda\}_{\lambda<\nu}$ is any transfinite sequence in $\mathbf{K}$.
(b) $\{x_\lambda\}_{\lambda<\nu}$is said to *satisfy property* (II) if for each $\epsilon > 0$, there exists a $\delta > 0$ (independent of $\{\alpha_\lambda\}_{\lambda<\nu}$ and μ) such that

$$\|\sum_{\chi\leq\mu} \alpha_\chi x_\chi\| > 1-\delta,\ \|\sum_{\chi<\nu} \alpha_\chi x_\chi\| = 1\ \Rightarrow \|\sum_{\mu+1\leq\chi<\nu} \alpha_\chi x_\chi\| \leq \epsilon,$$

where $\{\alpha_\lambda\}_{\lambda<\nu}$ is any transfinite sequence in $\mathbf{K}$ such that $\sum_{\chi<\nu} \alpha_\chi x_\chi$ converges.

These two properties are equivalent.

Justification.

(I)⇒ (II). Suppose that the transfinite basis $\{x_\lambda\}_{\lambda<\nu}$ satisfies property (I) but not property (II). Then, there exists an $\epsilon > 0$ such that for every $\delta > 0$, there is a transfinite sequence of scalars $\{\alpha_\lambda\}_{\lambda<\nu} \subset \mathbf{K}$ with $\sum_{\lambda<\nu} \alpha_\lambda x_\lambda$ convergent and an ordinal $\mu < \nu$ such that

$$\|\sum_{\chi\leq\mu} \alpha_\chi x_\chi\| > 1-\delta,\ \|\sum_{\chi<\nu} \alpha_\chi x_\chi\| = 1\ but\ \|\sum_{\mu+1\leq\chi<\nu} \alpha_\chi x_\chi\| > \epsilon.$$

Let $\gamma_c > 0$ be any arbitrary number. Put $\delta = \gamma_c(1+\gamma_c)^{-1}$ and
$\beta_\chi = \alpha_\chi(\|\sum_{\chi\leq\mu} \alpha_\chi x_\chi\|)^{-1}$, $(\chi < \nu)$. Then

$$\|\sum_{\chi\leq\mu} \beta_\chi x_\chi\| \ = \ 1$$

and

$$\begin{aligned} \|\sum_{\mu+1\leq\chi<\nu} \beta_\chi x_\chi\| \ &> \ \epsilon(\|\sum_{\chi\leq\mu} \alpha_\chi x_\chi\|)^{-1} \\ &\geq \ \epsilon/V_{\{x_\lambda\}_{\lambda<\nu}} \ = \ c \ > \ 0, \end{aligned}$$

where $V_{\{x_\lambda\}_{\lambda<\nu}}$ is the norm of the transfinite basis $\{x_\lambda\}_{\lambda<\nu}$. But

$$\|\sum_{\chi<\nu} \beta_\chi x_\chi\| \ < \ 1+\gamma_c.$$

Thus $\{x_\lambda\}_{\lambda<\nu}$ does not satisfy property (I).

(II) $\Rightarrow$ (I). Let $\{x_\lambda\}_{\lambda<\nu}$ satisfy property (II) but not property (I). Then, there exists a $c>0$ such that for every $\gamma_c>0$, there is a transfinite sequence of scalars $\{\alpha_\lambda\}_{\lambda<\nu}\subset \mathbf{K}$ and an ordinal $\mu<\nu$ such that

$$\|\sum_{\chi\le\mu}\alpha_\chi x_\chi\|=1,\ \|\sum_{\mu+1\le\chi<\nu}\alpha_\chi x_\chi\|\ge c \ but \ \|\sum_{\chi<\nu}\alpha_\chi x_\chi\|<1+\gamma_c.$$

We now show that there is an $\epsilon>0$ such that for no $\delta>0$, the relation in property (II) is satisfied. Let $0<\eta<1$ be arbitrary but fixed. Let $\delta>0$ be arbitrary such that $\delta\le\eta$. Put $\gamma_c=\delta(1-\delta)^{-1}$ and $\beta_\chi=\alpha_\chi(\|\sum_{\chi<\nu}\alpha_\chi x_\chi\|)^{-1}$.

Then

$$\|\sum_{\chi\le\mu}\beta_\chi x_\chi\| > 1-\delta \quad and \quad \|\sum_{\chi<\nu}\beta_\chi x_\chi\| = 1$$

But

$$\begin{aligned}\|\sum_{\mu+1\le\chi<\nu}\beta_\chi x_\chi\| &\ge c(\|\sum_{\chi<\nu}\alpha_\chi x_\chi\|)^{-1}\\ &> c(1-\delta)\\ &\ge c(1-\eta).\end{aligned}$$

Taking $\epsilon=c(1-\eta)$ our assertion is established.

Theorem 1. *If $\{x_\lambda\}_{\lambda<\nu}$ is a monotone transfinite basis in a uniformly convex Banach space E, then $\{x_\lambda\}_{\lambda<\nu}$ satisfies property (II) (and hence property (I)).*

Proof. Suppose on the contrary that $\{x_\lambda\}_{\lambda<\nu}$ does not satisfy property (II). Then, there exists an $\epsilon>0$ such that for every $\delta>0$ one can find $\{\alpha_\lambda^{(\delta)}\}_{\lambda<\nu}\subset\mathbf{K}$ with $\sum_{\lambda<\nu}\alpha_\lambda^{(\delta)}x_\lambda$ converging in E, and an ordinal number $\mu(\delta)<\nu$ such that

$$\|\sum_{\chi\le\mu(\delta)}\alpha_\chi^{(\delta)}x_\chi\| > 1-\delta, \qquad \|\sum_{\chi<\nu}\alpha_\chi^{(\delta)}x_\chi\| = 1$$

and

$$\|\sum_{\mu(\delta)+1\le\chi<\nu}\alpha_\chi^{(\delta)}x_\chi\| > \epsilon.$$

Then, in view of monotonicity of $\{x_\lambda\}_{\lambda<\nu}$, we have

$$1-\delta < \|\sum_{\chi\le\mu(\delta)}\alpha_\chi^{(\delta)}x_\chi\| \le \|\sum_{\chi<\nu}\alpha_\chi^{(\delta)}x_\chi\| = 1.$$

Since E is uniformly convex, there is a $\delta_0\equiv\delta_0(\epsilon)$, $0<\delta_0<1$, such that whenever $\|x\|\le 1$, $\|y\|\le 1$ and $\|x-y\|>\epsilon$, we have $\|\frac{x+y}{2}\|\le 1-\delta_0$. Writing

$$x_0 = \sum_{\chi\le\mu(\delta_0)}\alpha_\chi^{(\delta_0)}x_\chi \quad \text{and} \quad y_0 = \sum_{\chi<\nu}\alpha_\chi^{(\delta_0)}x_\chi,$$

we have $||x_0|| \le 1$, $||y_0|| = 1$ and $||x_0 - y_0|| > \epsilon$. Hence

$$\begin{aligned} 1-\delta_0 &\le ||\frac{x_0+y_0}{2}|| \\ &= ||\sum_{\chi \le \mu(\delta_0)} \alpha_\chi^{(\delta_0)} x_\chi + \frac{1}{2}\sum \sum_{\mu(\delta_0)+1 \le x < \nu} \alpha_\chi^{(\delta_0)} x_\chi|| \\ &\ge ||\sum_{\chi \le \mu(\delta_0)} \alpha_\chi^{(\delta_0)} x_\chi|| > 1-\delta_0. \end{aligned}$$

This is absurd.

Theorem 2. *If $\{x_\lambda\}_{\lambda<\nu} \subset E$ is a transfinite basis satisfying property (I), then $\{x_\lambda\}_{\lambda<\nu}$ is a boundedly complete transfinite basis.*

Proof. Let $\lambda_1 < \lambda_2 < \ldots < \lambda \le \nu$ be such that $\lim_{j\to\infty} \lambda_j = \lambda$ and $\{y_n\} \subset E$ be any (countable) bounded sequence such that

$$S_{\lambda_n}(y_j) = y_n, \qquad (n = 1, 2, \ldots, j; j \in N).$$

To complete the proof, it is enough to establish that $\lim_{n\to\infty} y_n = x (\in E)$ exists. Choose a sequence $\{n_k\}$ of positive integers such that

$$\lim_{k\to\infty} ||y_{n_k}|| = \overline{\lim}_{n\to\infty} ||y_n|| = B.$$

If $B = 0$, then $\lim_{n\to\infty} y_n = 0 (\in E)$ exists. If $B \ne 0$, we shall show that $\{y_{n_k}\}$ is a cauchy sequence. Suppose on the contrary that $\{y_{n_k}\}$ is not a cauchy sequence, then there exists a $\delta > 0$ and subsequences $\{y_{n_{k_j}}\}$, $\{y_{n_{p_j}}\}$ of $\{y_{n_k}\}$ with $n_{k_j} > n_{p_j}$, $(j \in N)$ such that

$$||y_{n_{k_j}} - y_{n_{p_j}}|| \ge \delta, \qquad (j \in \mathbf{N}).$$

Since

$$||\frac{y_{n_{k_j}} - y_{n_{p_j}}}{||y_{n_{p_j}}||}|| \ge \frac{\delta}{A} = c > 0,$$

where $A = \sup_n ||y_n|| < \infty$, it follows, in view of Property (I),that

$$\lim_{j\to\infty} ||y_{n_{k_j}}|| \ge \lim_{j\to\infty} ||y_{n_{p_j}}||(1+\gamma_c).$$

This gives $B \ge B(1+\gamma_c)$, which is impossible since $B \ne 0$.

Hence, $\{y_{n_k}\}$ is a cauchy sequence in E. Let $\lim_{k\to\infty} y_{n_k} = x$, where $x \in E$. Then

$$\begin{aligned} x = \lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty} S_{\lambda_{n_k}}(y_j) \\ &= S_\lambda(y_j) \\ &= \lim_{n\to\infty} S_{\lambda_n}(y_j), \qquad (j \in \mathbf{N}) \\ &= \lim_{n\to\infty} y_n. \end{aligned}$$

This completes the proof.

Corollary. *If $\{x_\lambda\}_{\lambda<\nu}$ is a monotone transfinite basis of a uniformly convex Banach Space E, then $\{x_\lambda\}_{\lambda<\nu}$ is boundedly complete transfinite basis of E.*

The converse of the Theorem 2 is not true.

Example. Let

$$\begin{aligned} E &= l^1[\nu],\ [\nu]=[1,\nu) \\ &= \{\{\alpha_\chi\}_{\chi<\nu} \subset K \mid \operatorname{card}\{\lambda : \alpha_\lambda \neq 0\} \leq \mathcal{N}_0\} \end{aligned}$$

where $\|\{\alpha_\lambda\}_{\lambda<\nu}\| = \sum_{\chi<\nu} \mid \alpha_\chi \mid < \infty$. Let $\{e_\lambda\}_{\lambda<\nu}$ be the natural transfinite basis of $l^1[\nu]$. Define a transfinite sequence $\{x_\lambda\}_{\lambda<\nu}$ in E by $x_1 = \frac{1}{2}(e_1+e_2)$, $x_2 = \frac{1}{2}(-e_1+e_2)$ and $x_\lambda = e_\lambda$ for $2 < \lambda < \nu$. Then, $\{x_\lambda\}_{\lambda<\nu}$ is a transfinite basis for $l^1[\nu]$, since for any $x = \{\alpha_\lambda\}_{\lambda<\nu} \in l^1[\nu]$, there is a unique transfinite sequence $\{\beta_\lambda\}_{\lambda<\nu}$ given by $\beta_1 = \alpha_1+\alpha_2$, $\beta_2 = -\alpha_1+\alpha_2$ and $\beta_\lambda = \alpha_\lambda$ for $2<\lambda<\nu$, satisfying

$$x = \sum_{\lambda<\nu} \beta_\lambda x_\lambda.$$

Furthermore, $\{x_\lambda\}_{\lambda<\nu}$ is boundedly complete. Indeed, if $\lambda_1 < \lambda_2 < \ldots < \lambda \leq \nu$ such that $\lim_{j\to\infty} \lambda_j = \lambda$ and $\{y_n\} \subset E$ is any (countable) bounded sequence satisfying

$$S_{\lambda_n}(y_j) = y_n, \qquad (n=1,2,\ldots,j;\ j\in N),$$

then

$$\begin{aligned} \sum_{\chi<\lambda_n} \alpha_\chi^{(j)} x_\chi &= S_{\lambda_n}(y_j) = y_n \\ &= \sum_{\chi<\nu} \alpha_\chi^{(n)} x_\chi, \qquad (n=1,2,\ldots,j;\ j\in N) \end{aligned}$$

where

$$y_j = \sum_{\chi<\nu} \alpha_\chi^{(j)} x_\chi, \qquad (j\in N).$$

This gives

$$\alpha_\chi^{(n)} = \begin{cases} \alpha_\chi^{(j)}; & \chi<\lambda_n \\ 0; & \chi \geq \lambda_n. \end{cases}$$

Thus

$$y_j = \sum_{n=1}^{j} \sum_{\lambda_{n-1}\leq\chi<\lambda_n} \alpha_\chi^{(n)} x_\chi, \qquad (j\in N;\ \lambda_0=1).$$

Put $\alpha_\chi = \alpha_\chi^{(n)}(\lambda_{n-1}\leq\chi<\lambda_n; n\in N, \lambda_0=1)$. Then

$$y_j = \sum_{n=1}^{j} \sum_{\lambda_{n-1}\leq\chi<\lambda_n} \alpha_\chi x_\chi, \qquad (j\in N)$$

Hence

$$\lim_{j\to\infty} y_j = \sum_{n=1}^{\infty} \sum_{\lambda_{n-1}\leq\chi<\lambda_n} \alpha_\chi x_\chi = \sum_{\chi<\nu} \alpha_\chi x_\chi.$$

On the other hand $\{x_\lambda\}_{\lambda<\nu}$ does not satisfy property (I) since, for $\beta_1 = \beta_2 = 1$ and $\beta_\lambda = 0,\ \lambda < \nu,\ \lambda \neq 1, 2$; we have $||x_1|| = 1$ and $||x_2 + \sum_{3\leq\chi<\nu} 0 \cdot x_\chi|| = 1$. But

$$||x_1 + (x_2 + \sum_{3\leq\chi<\nu} 0 \cdot x_\chi)|| = ||e_2|| = 1.$$

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