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Suberesolving codes

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Abstract. We show that any right continuing factor code with retract 0 into an irreducible shift of finite type is right eresolving, and we give some sufficient conditions for a right eresolving almost everywhere code being right eresolving everywhere. Suberesolving codes as a generalization of eresolving codes have been introduced and we determine some shift spaces which preserved by suberesolving codes. Also, we show that any bi-eresolving (resp. bi-suberesolving) code on an irreducible shift of finite type (resp. a synchronized system) is open (resp. semi-open) and any right suberesolving code on a synchronized system is right continuing almost everywhere.

Keywords. shift of finite type, sofic, synchronized, coded, open, eresolving, continuing

1 Introduction

Sliding block codes form bases for more studies on other topics in Symbolic Dynamics such as hidden Markov chains. They are divided into two categories: finite-to-one codes and infinite-to-one codes. The simplest class in the first category is the class of right resolving codes. Many things are known about right resolving codes [3, 8]. A right eresolving code is a natural dual version of a right resolving code.

A 1-block factor code $\varphi : X \to Y$ is said to be *right eresolving* if, whenever $ab \in \mathcal{B}_2(Y)$ and $\overline{a} \in \varphi^{-1}(a)$, there is $\overline{b} \in \varphi^{-1}(b)$ such that $\overline{a}\,\overline{b} \in \mathcal{B}_2(X)$. In [2], right continuing codes with retract 0 called right eresolving. These two definitions are different, but they are equivalent when X is SFT. In this paper, right eresolving refers to the first definition.

Eresolving codes are subject of interest in the field of symbolic dynamics and ergodic theory. These codes are less well understood than resolving codes are, but some nice properties of such codes are known. For example, eresolving codes between SFTs work nicely with Markov measures; if $\varphi : X \to Y$ is eresolving between irreducible shifts of finite type and $k \in \mathbb{N}$, then every k-step Markov measure on Y lifts to a k-step Markov Measure on X, so they are Markovian. The main point about these is that infinite-to-one eresolving codes between subshifts of finite type lift each Markov measure to uncountably many Markov measures [5]. In [9], Marcus, Petersen and Williams construct infinite-to-one codes through which no Markov measure lifts to a Markov measure.

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Also, in [4], it is proved that any factor code between irreducible sofic shifts can be lifted to a right eresolving factor code between irreducible SFTs via some right resolving factor maps.

A main task in this note is to define suberesolving codes as a generalization of eresolving codes and investigate some properties of eresolving codes and suberesolving codes. A summary of our results is as follows. In Section 3, we show that any right continuing factor code with retract 0 into an irreducible shift of finite type is right eresolving (Theorem 3.1). Theorem 3.2 implies that a right eresolving almost everywhere factor code $\varphi : X \to Y$, where X is an irreducible sofic shift and Y is an irreducible SFT, is right eresolving (everywhere). Then, we prove that any bi-eresolving code on an irreducible shift of finite type is open. In Section 4, first we define subereolving codes and then, we determine some shift spaces which preserved by suberesolving codes (Corollary 4.2). Then, by Theorem 4.5 (resp. 4.7) we show that and any bi-suberesolving (resp. right suberesolving) code on a synchronized system is semi-open (resp. right continuing almost everywhere). Finally, in Theorem 4.8, we give conditions on suberesolving and bi-continuing codes to be semi-open with a uniform lifting length.

2 Background and Notations

The notations has been taken from [8] and the proofs of the claims in this section can be found there. Let $\mathcal{A}$ be a non-empty finite set of alphabet. The full $\mathcal{A}$-shift, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols over $\mathcal{A}$. A block (or word) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The *shift map* is the map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(\{x_i\}) = \{y_i\}$ where $y_i = x_{i+1}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *full shift*. Let $\mathcal{F}$ be a set of words over $\mathcal{A}$. Define $X_{\mathcal{F}}$ to be the subset of $\mathcal{A}^{\mathbb{Z}}$ which do not contain any word in $\mathcal{F}$. A *shift space* or a *subshift* X is a closed subset of $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some set $\mathcal{F}$. A subshift X is a *shift of finite type* (SFT) if $X = X_{\mathcal{F}}$ for a finite set $\mathcal{F}$. Shifts of finite type are sometimes called topological Markov shifts.

Denote by $\mathcal{B}_n(X)$ the set of all admissible n-words and let

$$\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$$

be the *language* of X. The *cylinder* ${}_l[u]$ is the set $\{x \in X : x_{[l,l+|u|-1]} = u\}$ where $u \in \mathcal{B}(X)$. For $l \geq 0$ and $|u| = 2l+1$, ${}_{-l}[u]$ is a *central* $2l+1$ cylinder.

A shift space X is *irreducible* if for every ordered pair of words $u, v \in \mathcal{B}(X)$, there is a word $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$. A point x in a shift space X is called *left-transitive* if every word in X appears in $x_{(-\infty,0]}$ infinitely often to the left and it is *doubly transitive* if every word in X appears in x infinitely often to the left and to the right.

Let $\mathcal{A}$ and $\mathcal{D}$ be the set of alphabets and X a shift space over $\mathcal{A}$. For $m, n \in \mathbb{Z}$ with $-m \leq n$, define the $(m+n+1)$-*block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ by

$$y_i = \Phi(x_{i-m}x_{i-m+1}...x_{i+n}) = \Phi(x_{[i-m,i+n]}) \tag{2.1}$$

where $y_i \in \mathcal{D}$. The map $\varphi = \Phi_\infty^{[-m,n]} : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $y = \varphi(x)$ with y_i given by (2.1) is called the *code* with memory m and anticipation n induced by Φ. If $m = n = 0$, then φ is called a 1-*block code* and we set $\varphi = \Phi_\infty$. Given a code $\varphi : X \to Y$, we can recode X to a conjugate shift $\overline{X}$, so that the corresponding sliding block code $\overline{\varphi} : \overline{X} \to Y$ is a 1-block code. This process, called 'recoding φ to a 1-block code' [8, Proposition 1.5.12]. An onto code is called a *factor code* and if it is also invertible, it is called *conjugacy*.

A code $\varphi : X \to Y$ is called *finite-to-one* if there exists a positive integer M such that $|\varphi^{-1}(y)| \leq M$ for any $y \in Y$.

A 1-block code $\varphi = \Phi_\infty : X \to Y$ is *right resolving* if whenever $ab, ac \in \mathcal{B}_2(X)$ with $\Phi(b) = \Phi(c)$, then $b = c$. A *left resolving* code is defined similarly. If φ is both left and right resolving, it is called *bi-resolving*.

A pair $x, \overline{x}$ of points in X is *left asymptotic* if there is an integer N such that $x_{(-\infty,N]} = \overline{x}_{(-\infty,N]}$. A code $\varphi : X \to Y$ is called *right continuing* if whenever $x \in X$, $y \in Y$ and $\varphi(x)$ is left asymptotic to y, then there exists at least one $\overline{x} \in X$ such that $\overline{x}$ is left asymptotic to x and $\varphi(\overline{x}) = y$. A *left continuing* code is defined similarly. If φ is both left and right continuing, it is called *bi-continuing*. An integer $n \in \mathbb{Z}^+$ is called a *(right continuing) retract* of a right continuing code $\varphi : X \to Y$ if, whenever $x \in X$ and $y \in Y$ with $\varphi(x)_{(-\infty,0]} = y_{(-\infty,0]}$, we can find $\overline{x} \in X$ such that $\varphi(\overline{x}) = y$ and $x_{(-\infty,-n]} = \overline{x}_{(-\infty,-n]}$ [7].

A code $\phi : X \to Y$ is called *right continuing almost everywhere (a.e.)* if whenever x is left transitive in X and $\phi(x)$ is left asymptotic to a point $y \in Y$, then there exists $\overline{x} \in X$ such that $\overline{x}$ is left asymptotic to x and $\phi(\overline{x}) = y$. Similarly we have the notions called left continuing a.e. and bi-continuing a.e.

Let G be a directed graph and $\mathcal{V}$ (resp. $\mathcal{E}$) the set of its vertices (resp. edges) which is supposed to be countable. The edge shift X_G is a shift space consisting of all bi-infinite sequences of edges from $\mathcal{E}$. Any SFT is conjugate to an edge shift. A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a directed graph with the edges set $\mathcal{E}$ and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ its labeling. Given a labeled graph $\mathcal{G}$, a shift space

$$X_{\mathcal{G}} = \text{closure}\{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \overline{\mathcal{L}_\infty(X_G)}$$

is defined and $\mathcal{G}$ is called a *cover* of $X_{\mathcal{G}}$. When G is finite, $X_{\mathcal{G}} = \mathcal{L}_\infty(X_G)$ is called a *sofic shift*. Every shift of finite type is sofic.

A labeled graph $\mathcal{G}$ is called *right resolving* if for any vertex I of G the edges starting at I carry different labels. A *left resolving* cover is defined similarly. If $\mathcal{G}$ is both left and right resolving, it is called *bi-resolving*.

Let X be an irreducible sofic shift. A *Fischer cover* of X is a right resolving cover having the fewest vertices among all right resolving covers of X.

A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever $uv, vw \in \mathcal{B}(X)$, then $uvw \in \mathcal{B}(X)$. An irreducible shift space X is *synchronized* if it has a synchronizing word. A shift space X has *specification with variable gap length* (SVGL) if there exists $N \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(X)$, there exists $w \in \mathcal{B}(X)$ with $uwv \in \mathcal{B}(X)$ and $|w| \leq N$. A shift that has specification with variable gap length is synchronized.

For $x \in \mathcal{B}(X)$, call $x_- = (x_i)_{i<0}$ (resp. $x_+ = (x_i)_{i \in \mathbb{Z}^+}$) the *left (resp. right) infinite x-ray*. Let $X^+ = \{x_+ : x \in X\}$. The follower set of x_- (resp. $m \in \mathcal{B}(X)$) is defined as $\omega_+(x_-) = \{x_+ \in X^+ : x_- x_+ \text{ is a point in } X\}$ (resp. $\omega_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$). An irreducible shift space X is called *half-synchronized*, if there is a word $m \in \mathcal{B}(X)$ and a left-transitive point $x \in X$ such that $x_{[-|m|+1,0]} = m$ and $\omega_+(m) = \omega_+(x_{(-\infty,0]})$. Such an m is called a *half-synchronizing word* for X. A synchronizing word is a half-synchronizing word. So a synchronized system is half-synchronized [6].

3 Eresolving Codes

Definition 1. Let $\varphi = \Phi_\infty : X \to Y$ be a 1-block factor map between shift spaces. Then, φ is said to be *right eresolving* if, whenever $ab \in \mathcal{B}_2(Y)$ and $\overline{a} \in \Phi^{-1}(a)$, there is $\overline{b} \in \varphi^{-1}(b)$ such that

$\overline{a}\,\overline{b} \in \mathcal{B}_2(X)$. Left eresolving and bi-eresolving can be defined similarly.

Remark 1. By [2], if φ is right continuing with retract 0, then it is said to be right eresolving. In this paper, right eresolving refers to Definition 1. In [11, Proposition 2.1], Yoo showed that if $\varphi : X \to Y$ is a factor code between SFTs, these two definitions are equivalent, but differ when X is merely sofic: With Definition 1 we may have right eresolving factor maps over sofic shifts which are not right continuing. Now in Theorem 3.1, we delete the assumption that X is SFT and we show that right continuing with retract 0 gives right eresolving.

Theorem 3.1. *Let $\varphi = \Phi_\infty : X \to Y$ be a factor code and Y be an irreducible SFT. If φ is right continuing with retract* 0*, then it is right eresolving.*

Proof. Without loss of generality, we can assume that Y is an edge shift. Let $ab \in \mathcal{B}_2(Y)$ and $\overline{a} \in \Phi^{-1}(a)$. Also, let $y \in \mathcal{B}(Y)$ and $x \in \mathcal{B}(X)$ such that $y_0 y_1 = ab$ and $x_0 = \overline{a}$. So $\varphi(x)_0 = a = y_0$. Since every symbol in $\mathcal{B}(Y)$ is synchronizing, the point $\overline{y}$ given by

$$\overline{y}_i = \begin{cases} \varphi(x)_i & i < 0, \\ y_i & i \geq 0. \end{cases}$$

is a point in Y. Since φ is right continuing with retract 0 and $\varphi(x)_{(-\infty,0]} = \overline{y}_{(-\infty,0]}$, there is $\overline{x} \in X$ such that $\overline{x}_{(-\infty,0]} = x_{(-\infty,0]}$ and $\varphi(\overline{x}) = \overline{y}$. So $\overline{x}_1 \in \Phi^{-1}(b)$ and $\overline{a}\overline{x}_1 \in \mathcal{B}_2(X)$.

□

Definition 2. let X be an irreducible sofic shift with the Fischer cover $\mathcal{G} = (G, \mathcal{L})$ and $\varphi : X \to Y$ be 1-block, we say that φ is right eresolving almost everywhere if φ is right continuing almost everywhere and $\varphi \circ \mathcal{L}_\infty$ is right eresolving.

The following theorem shall be seen as the dual of [3, Proposition 4.10], which states that a right closing almost everywhere factor code from an SFT onto a sofic shift is right closing everywhere.

Theorem 3.2. *A right eresolving almost everywhere factor code $\varphi = \Phi_\infty : X \to Y$, where X is an irreducible sofic shift and Y is an irreducible SFT, is right eresolving (everywhere).*

Proof. Let $\mathcal{G} = (G, \mathcal{L})$ be the Fischer cover of X. By definition, $\varphi : X \to Y$ is right continuing a.e. and $\varphi \circ \mathcal{L}_\infty$ is right eresolving. Since X_G and Y are SFT, $\varphi \circ \mathcal{L}_\infty$ is right continuing with retract 0. Therefore, φ is right continuing with retract 0. Then, by Theorem 3.1, φ is right eresolving.

□

Theorem 3.3. *A right eresolving factor map $\varphi = \Phi_\infty : X \to Y$, where X is an irreducible SFT, is right continuing.*

Proof. Let $x \in X$, $y \in Y$ and $\phi(x)$ is left asymptotic to y. Without loss of generality, we can assume that X is an edge shift and $\varphi(x)_{(-\infty,0]} = y_{(-\infty,0]}$. Then, we have $y_0 y_1 \in \mathcal{B}_2(Y)$ and $x_0 \in \varphi^{-1}(y_0)$. Since φ is right eresolving, there is $\overline{x}_1 \in \Phi^{-1}(y_1)$ such that $x_0 \overline{x}_1 \in \mathcal{B}_2(X)$.

Now $y_1 y_2 \in \mathcal{B}_2(Y)$ and $\overline{x}_1 \in \Phi^{-1}(y_1)$. Therefore, there is $\overline{x}_2 \in \Phi^{-1}(y_2)$ such that $\overline{x}_1 \overline{x}_2 \in \mathcal{B}_2(X)$. Since X is an edge shift, $x_0 \overline{x}_1 \overline{x}_2 \in \mathcal{B}(X)$. Then, by repeating this process, we get $\overline{x} \in X$ such that $\overline{x}_{(-\infty,0]} = x_{(-\infty,0]}$, $\overline{x}_{[1,\infty)} = \overline{x}_1 \overline{x_2} \cdots$ and $\varphi(\overline{x}) = y$. So φ is right continuing. □

J. Yoo [12] showed that a right continuing factor of a shift of finite type is also of finite type, so by Theorem 3.3 we have:

Corollary 3.4. *Let X be a sofic shift with the cover $\mathcal{G} = (G, \mathcal{L})$. If $\mathcal{L}_\infty : X_G \to X$ is eresolving, then, X is SFT.*

A map $\varphi : X \to Y$ is called *open* if images of open sets are open and is called *semi-open* if images of open sets have non-empty interior.

Theorem 3.5. *Let $\varphi = \Phi_\infty : X \to Y$ be an bi-eresolving code and X be SFT. Then, φ is open.*

Proof. Since φ is bi-eresolving, for any $a \in \mathcal{B}_1(X)$, we have $[\Phi(a)] = \varphi([a])$. So φ is open. □

4 Properties of suberesolving codes

We extend Definition 1 as follow.

Definition 3. Let $\varphi = \Phi_\infty : X \to Y$ be a 1-block factor map between shift spaces. Then, φ is said to be *right suberesolving* if, whenever $uv \in \mathcal{B}(Y)$ and $\overline{u} \in \Phi^{-1}(u)$, there is $\overline{v} \in \Phi^{-1}(v)$ such that $\overline{u}\,\overline{v} \in \mathcal{B}(X)$. Left suberesolving and bi-suberesolving can be defined similarly.

It is clear that any right suberesolving code is right eresolving. But Example 1 shows that one right eresolving code may not be right suberesolving. However, a code on an irreducible SFT is right eresolving if and only if it is right suberesolving.

Example 1. Let X be a shift space on the alphabet $\{1, \overline{1}, 2, 3\}$ defined by forbidding $\{\overline{1}2^n3 : n \geq 1\}$, and let Y be the full 3-shift $\{1, 2, 3\}^{\mathbb{Z}}$. Define $\varphi = \Phi_\infty : X \to Y$ by letting $\Phi(\overline{1}) = 1$ and $\Phi(a) = a$ for all $a \neq \overline{1}$.

Then, φ is right eresolving, but it is not suberesolving. Because if $u = 12$, $v = 3$ and $\overline{u} = \overline{1}2$, there is not any $\overline{v} \in \Phi^{-1}(v)$ such that $\overline{u}\,\overline{v} \in \mathcal{B}_2(X)$.

Note that a suberesolving code may not be finite-to-one. Let $\varphi : \{0, 1, 2\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ be given by the 1-block map Φ, where $\Phi(0) = 0$, $\Phi(1) = 1$ and $\Phi(2) = 1$. Then, φ is suberesolving. But if $y \in \{0, 1, 2\}^{\mathbb{Z}}$ contains only finitely many 1's, then $\varphi^{-1}(y)$ is finite, while if y contains infinitely many 1's, then $\varphi^{-1}(y)$ is infinite.

By Definition 3, we have:

Theorem 4.1. *Let $\varphi = \Phi_\infty : X \to Y$ be a bi-suberesolving code. Then, the image of any synchronizing word in X by Φ is a synchronizing word in Y.*

Corollary 4.2. *Let $\varphi : X \to Y$ be a bi-suberesolving code. If X is synchronized (resp. SFT), then Y is also.*

Proof. By Theorem 4.1, the proof in the case synchronized is obvious. Now let X be SFT. So there is an $N \geq 0$ such that X is N-step. So by [8, Theorem 3.4.17], all N-words are synchronizing. Therefore, Theorem 4.1 implies that any word in $\mathcal{B}_N(Y)$ is synchronizing and hence, Y is SFT. □

Theorem 4.3. *Let $\varphi : X \to Y$ be a right suberesolving code. If X is half-synchronized, then Y is also.*

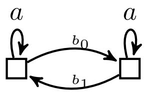

Figure 1: The cover of X in Example 2.

Proof. If X is half-synchronized, there is a word $m \in \mathcal{B}(X)$ and a left-transitive point $x \in X$ such that $x_{[-|m|+1,0]} = m$ and $\omega_+(x_{(-\infty,0]}) = \omega_+(m)$. Now let $y = \varphi(x) \in Y$ and $\overline{m} = \Phi(m)$. Then, y is a left transitive point and we have $\omega_+(y_{(-\infty,0]}) = \omega_+(\overline{m})$. So Y is half-synchroniozed. □

Given a shift space X, Thomsen [10] defined the derived shift space ∂X. Let $\mathrm{Per}(X)$ be the set of periodic points of the shift space X and let $R(X) = \overline{\mathrm{Per}(X)}$. The *derived shift space* of X is defined by

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\}.$$

The derived shift space is a subshift of X. By Theorem 4.1, we have:

Theorem 4.4. *Let $\varphi : X \to Y$ be a bi-suberesolving code. Then, $\varphi^{-1}(\partial Y) \subseteq \partial X$.*

Theorem 4.4 gives another way to prove that any irreducible bi-eresolving factor of a SFT is SFT. Because the derived shift space of a SFT is empty set [10].

The following example shows that the inclusion in Theorem 4.4 is not always an equality.

Example 2. Let X be the sofic shift obtained by identifying two fixed points in the full 2-shift, $Y = \{a, b\}^{\mathbb{Z}}$ the full 2-shift, and $\varphi : X \to Y$ the subscript dropping code (see Figure 1). Note that φ is a bi-suberesolving code and $\partial Y = \emptyset$, while $\partial X = \{a^\infty\}$.

Theorem 4.5. *Let $\varphi = \Phi_\infty : X \to Y$ be an bi-suberesolving code and X be synchronized. Then, φ is semi-open.*

Proof. It is sufficient to prove that for any $a \in \mathcal{B}_1(X)$, $\varphi([a])$ has non-empty interior. a can be extended on the right to a synchronizing word $av \in \mathcal{B}(X)$. Since φ is bi-suberesolving, $[\Phi(av)] = \varphi([av])$ and since $\varphi([av]) \subseteq \varphi([a])$, $[\Phi(av)] \subseteq \varphi([a])$. Obviously, the proof holds for any cylinder set and so φ is semi-open. □

Theorem 4.6. *Let $\varphi = \Phi_\infty : X \to Y$ be a right suberesolving and left continuing code with retract 0 and X be half-synchronized. Then, φ is semi-open.*

Proof. Let $m \in \mathcal{B}(X)$ and $x \in X$ be a left transitive point such that $x_{[-|m|+1,0]} = m$ and $\omega_+(x_{(-\infty,0]}) = \omega_+(m)$ and also, $a \in \mathcal{B}_1(X)$. Since X is irreducible, without loss of generality, we may assume that $ma \in \mathcal{B}(X)$. We claim that $[\Phi(ma)] \subseteq \varphi([a])$.

Let $y \in [\Phi(ma)]$. We may assume that $y = \cdots y_{-2}y_{-1}\Phi(ma)y_{|m|+1}\cdots$. Since φ is right suberesolving and X is compact, there is $z_+ \in X^+$ such that $\overline{x} = x_{-}az_+ \in X$ and $\varphi(z_+) = y_{[2,+\infty)}$. Now $\varphi(\overline{x})$ and y are right asymptotic and since φ is left continuing, so there is at least one $\overline{\overline{x}} \in X$ such that $\overline{\overline{x}}_{[0,+\infty)} = \overline{x}_{[0,+\infty)}$ and $\varphi(\overline{\overline{x}}) = y$. Since, $\overline{\overline{x}} \in [a]$, we have $[\Phi(ma)] \subseteq \varphi([a])$. Obviously, the proof holds for any cylinder set and so φ is semi-open. □

Theorem 4.7. *Suppose that $\varphi = \Phi_\infty : X \to Y$ is a right suberesolving code and X is synchronized. Then, φ is right continuing a.e..*

Proof. Let $x \in X$ be a left transitive point and $\varphi(x)$ is left asymptotic to a point $y \in Y$. Without loss of generality, we can assume that $\varphi(x)_{(-\infty,0]} = y_{(-\infty,0]}$. Also, let $m \in \mathcal{B}(X)$ be a synchronizing word. Since x is left transitive, there is a negative integer n such that $x_{n-|m|}x_{n-|m|+1}\cdots x_{n-1} = m$. Since φ is right suberesolving, there is a $z_+ \in X^+$ such that $mz_{[n,\infty)} \in X^+$ and $\varphi(mz_{[n,\infty)}) = \Phi(m)y_{[n,\infty)}$. Now since m is a synchronizing word, $\overline{x} = x_{(-\infty,n-|m|-1]}mz_{[n,\infty)} \in X$ and $\varphi(\overline{x}) = y$. So φ is right continuing a.e.. □

Let us state an equivalent definition for a semi-open code.

Lemma 4.1. *[1] A code $\phi : X \to Y$ between shift spaces is semi-open if for each $l \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that the image of a central $2l+1$ cylinder in X contains a central $(2k+1)$ cylinder in Y.*

Let φ be a semi-open code. We say that ϕ has a *uniform lifting length* if for each $l \in \mathbb{N}$, there exists k satisfying the above property such that $\sup_l |k-l| < \infty$.

Theorem 4.8. *Let $\varphi = \Phi_\infty : X \to Y$ be a right suberesolving code and Y a SVGL subshift. If φ is bi-continuing with a bi-retract, then it will be semi-open with a uniform lifting length.*

Proof. Suppose that $\varphi = \Phi_\infty$ is bi-continuing with a bi-retract $n \in \mathbb{N}$. Since Y is a SVGL subshift, there exists $k \in \mathbb{N}$ such that for all $u, v \in \mathcal{B}(Y)$, there exists $w \in \mathcal{B}(Y)$ with $uwv \in \mathcal{B}(Y)$ and $|w| \le k$ and also, there exists a synchronizing word in $\mathcal{B}(Y)$. Without loss of generality, we assume that $b \in \mathcal{B}_1(Y)$ is a synchronizing word for Y. Let $l \ge 0$ and $u \in \mathcal{B}_{2l+1}(X)$. There exists $w \in \mathcal{B}(Y)$ with $\Phi(u)wb \in \mathcal{B}(Y)$ and $|w| \le k$. Since φ is right suberesolving, there is $va \in \Phi^{-1}(wb)$ such that $uva \in \mathcal{B}(X)$.

Choose $x \in {}_l[u]$ to be a left-transitive point such that $x_{[-l,l+|v|+1]} = uva$ and pick a point $y \in Y$ with $y_{[-\alpha,\alpha]} = \varphi(x)_{[-\alpha,\alpha]}$ where $\alpha = l+n+k+1$. Since b is synchronizing and $y_{l+|w|+1} = b$, the point $\hat{y}$ given by

$$\hat{y}_i = \begin{cases} \varphi(x)_i & i \le 0, \\ y_i & i > 0. \end{cases}$$

is a left transitive point in Y. Since n is a right continuing retract and $\varphi(x)_{(-\infty,\alpha]} = \hat{y}_{(-\infty,\alpha]}$, there is $\overline{x} \in X$ such that $\overline{x}_{(-\infty,l+k+1]} = x_{(-\infty,l+k+1]}$ and $\varphi(\overline{x}) = \hat{y}$. We have $\varphi(\overline{x})_{[-\alpha,\infty)} = \hat{y}_{[-\alpha,\infty)} = y_{[-\alpha,\infty)}$. By the fact that n is also a left continuing retract, there is $\overline{\overline{x}} \in X$ such that $\overline{\overline{x}}_{[-l-k-1,\infty)} = \overline{x}_{[-l-k-1,\infty)}$ and $\varphi(\overline{\overline{x}}) = y$. Note that

$$\overline{\overline{x}}_{[-l-k-1,\,l+k+1]} = \overline{x}_{[-l-k-1,\,l+k+1]} = x_{[-l-k-1,\,l+k+1]}$$

which means that $\overline{\overline{x}}$ is in ${}_l[u]$ and so, $y = \varphi(\overline{\overline{x}}) \in \varphi({}_l[u])$. Therefore, $[\varphi(x)_{[-\alpha,\alpha]}] \subseteq \varphi({}_l[u])$ and φ is semi-open with uniform lifting length.

□

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