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PROPERTIES OF DOMAINLIKE RINGS

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Abstract. In this paper we will examine properties of and relationships between rings that share some properties with integral domains, but whose definitions are less restrictive. If R is a commutative ring with identity, we call R a *domainlike* ring if all zero-divisors of R are nilpotent, which is equivalent to (0) being primary. We exhibit properties of domainlike rings, and we compare them to présimplifiable rings and (hereditarily) strongly associate rings. Further, we consider idealizations, localizations, zero-divisor graphs, and ultraproducts of domainlike rings.

1. Introduction

Let R be a commutative ring with identity with total quotient ring $T(R)$, group of units $U(R)$, set of zero-divisors $Z(R)$, and Jacobson radical $J(R)$. If $A \subseteq R$, we use A^* to denote the nonzero elements of A. We call a ring R local if R is Noetherian and has a unique maximal ideal, and R is quasi-local if R has a unique maximal ideal but is not necessarily Noetherian.

For $a, b \in R$, we define three associate relations found in [2]. We say a and b are *associate*, denoted $a \sim b$, if $a|b$ and $b|a$, or equivalently, if $(a) = (b)$. We say a and b are *strongly associate*, denoted $a \approx b$, if there exists a $u \in U(R)$ such that $a = ub$. We say a and b are *very strongly associate*, denoted $a \cong b$, if $a \sim b$ and either $a = 0$, or $a = rb$ implies $r \in U(R)$. A ring R is a *strongly associate ring* if $a \sim b$ implies $a \approx b$. A ring R is a *hereditarily strongly associate ring* if every subring of R is a strongly associate ring. The study of strongly associate rings was begun by Kaplansky in [24] and has been further studied in [2], [4] and [30].

Recall that a ring R is called présimplifiable if $xy = x$ for x, $y \in R$ implies that either $x = 0$ or $y \in U(R)$. Présimplifiable rings were introduced by Bouvier in [13] - [17] and later studied by D.D. Anderson et al. in [2], [4] and [5]. The following theorem shows how the property of présimplifiable is related to the types of associate elements defined above.

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Theorem 1. [2, Theorem 1] *For a commutative ring R the following are equivalent.*

(1) $a \sim b \Rightarrow a \cong b$ *for all* $a, b \in R$;

(2) $a \approx b \Rightarrow a \cong b$ *for all* $a, b \in R$;

(3) R *is présimplifiable*;

(4) $Z(R) \subseteq J(R)$;

As an example, it is easy to check that $\mathbb{Z}_n$ is présimplifiable if and only if $n = p^m$, where p is some prime. Hence, $\mathbb{Z}_n$ is présimplifiable if and only if $\mathbb{Z}_n$ is local. Also, if a ring R is quasi-local, then R is présimplifiable, since $J(R) = M$, the unique maximal ideal of R, and thus $Z(R) \subseteq M = J(R)$.

It is straightforward to show that if R is présimplifiable, then R is a strongly associate ring. The converse is false, since a direct product of strongly associate rings is strongly associate [2, Theorem 3], but a présimplifiable ring has no nontrivial idempotents and hence is indecomposable. Also, any integral domain or any quasi-local ring is présimplifiable and hence a strongly associate ring.

A présimplifiable ring R is defined in terms of a weakened cancellation property and exhibits some of the same properties as an integral domain. The definition below, introduced by Spellman et al. in [30] and explored by Anderson et al. in [2], is a type of ring sharing more properties with integral domains than présimplifiable rings.

Definition 2. A ring R is *domainlike* if $Z(R) \subseteq \mathrm{nil}(R)$, the nilradical of R.

It is straightforward to verify that (0) is primary in R if and only if R is domainlike. A classic and elementary result of commutative ring theory is that an ideal P is prime if and only if R/P is an integral domain. The reader can quickly establish a parallel result with domainlike rings; namely, an ideal Q is primary if and only if R/Q is domainlike.

By Theorem 1, if R is domainlike, then R is also présimplifiable. However, R being présimplifiable does not imply that R is domainlike, as the following example shows.

Example 3. Let $R = K\left[[x, y]\right] / (x)(x, y)$. Then R is local and hence is présimplifiable. However, R is not domainlike, since $Z(R) = (\overline{x}, \overline{y})$, while $\mathrm{nil}(R) = (\overline{x})$.

To summarize, we have the following implications.

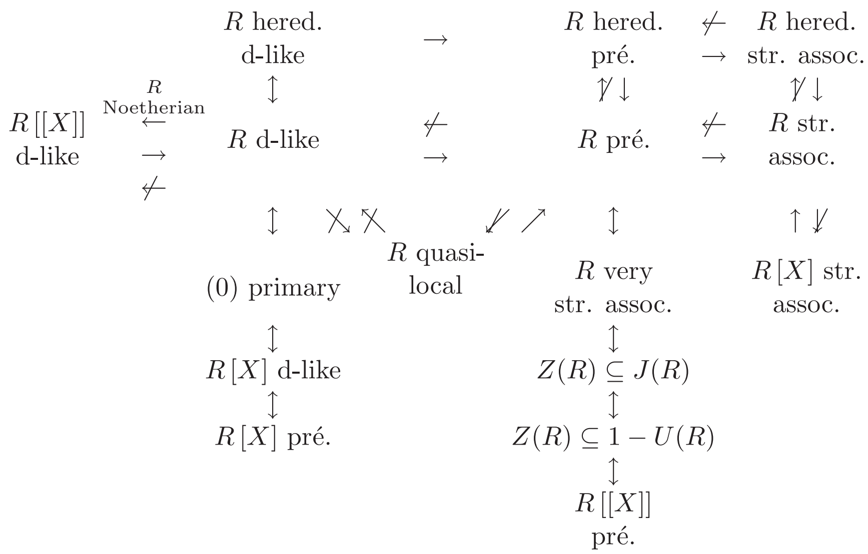
$$R \text{ is quasi-local} \Rightarrow R \text{ is présimplifiable} \Rightarrow R \text{ is strongly associate}$$
$$R \text{ is domainlike} \Rightarrow R \text{ is présimplifiable} \Rightarrow R \text{ is strongly associate}$$

However, there is no implication between domainlike and quasi-local. Example 3 shows that a quasi-local (in fact, local) ring need not be domainlike. Further, $\mathbb{Z}$ is domainlike, présimplifiable, and hereditarily strongly associate, but not quasi-local. It is also of interest to note that a domainlike ring may be neither Noetherian nor quasi-local. For example, $R = \mathbb{Z}[2X, 2X^2, 2X^3, \ldots]$ is domainlike, and R is not Noetherian, since the ideal

$P = (2X, 2X^2, 2X^3, \ldots)$ cannot be finitely generated. Further, $(2, 2X, 2X^2, 2X^3, \ldots)$ is maximal, as is $(3, 2X, 2X^2, 2X^3, \ldots)$. Thus, R is domainlike, not quasi-local, and not Noetherian.

If R is hereditarily strongly associate, then R need not be présimplifiable, domainlike, nor quasi-local. For example, the reader can easily verify that $\mathbb{Z} \times \mathbb{Z}$ is hereditarily strongly associate. However, a direct product of présimplifiable rings is never présimplifiable, and a direct product of domainlike rings is never domainlike. In addition, being hereditarily strongly associate is not preserved by direct products or subdirect products [30, Remark 2].

The diagram below offers a summary of the results mentioned above, as well as results from [2].



Here are some remaining questions concerning the above diagram.

(1) If R is strongly associate, is $R[[X]]$ strongly associate? [2, Question 21].

(2) If $R[X]$ is strongly associate, is $R[X]$ présimplifiable?

(3) If R is hereditarily présimplifiable (every subring of R is présimplifiable), is R domainlike?

In Section 2 we will focus on domainlike rings and compare them to présimplifiable rings and (hereditarily) strongly associate rings. In Section 3 we will explore idealizations and localizations of domainlike rings. We then consider zero-divisor graphs and ultraproducts of domainlike rings in Sections 4 and 5.

2. Domainlike properties and Présimplifiable rings

It is of interest to note that if R is domainlike, then so is the total quotient ring $T(R)$. To see this, assume R is domainlike. Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in T(R)$ with $\frac{r_1}{s_1} \neq 0$ and assume $\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{0}{1}$. There exists an $\overline{s} \in reg(R)$ such that $\overline{s}(r_1 r_2 - 0) = 0$. Thus, $r_1 r_2 = 0$. If $r_1 \neq 0$, then $r_2^n = 0$ for some n, since (0) is primary in R. Hence, $(\frac{r_2}{s_2})^n = \frac{0}{1}$, and so (0) is primary in $T(R)$.

It is also straightforward to see that if R is domainlike, then $R/\text{nil}(R)$ is an integral domain. The converse however is false. As in Example 3, take $R = K\,[[x,y]]\,/\,(x)\,(x,y)$. We see R is not domainlike, but $\text{nil}(R) = (\overline{x})$ is prime.

The following result is another interesting property of domainlike rings.

Lemma 4. *If R is domainlike, then $Z(R)$ is the unique minimal prime ideal of R.*

Proof. This follows when R is domainlike, since if P is a prime ideal, then $\text{nil}(R) \subseteq P$.

In particular, the above lemma provides an alternate proof of the fact that if R is domainlike, then $R/\text{nil}(R)$ is an integral domain. In general, a domainlike ring is not necessarily an integral domain, but as the next result shows for rings of the form $R/\text{rad}(I)$, where I is an ideal of R and $\text{rad}(I) = \{r \in R \mid r^n \in I \text{ for some } n > 0\}$, the two concepts are equivalent.

Lemma 5. *For a ring R and an ideal I of R, $R/\text{rad}(I)$ is domainlike if and only if $R/\text{rad}(I)$ is an integral domain. In particular, $R/\text{nil}(R)$ is domainlike if and only if $R/\text{nil}(R)$ is an integral domain.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Suppose $\overline{ab} = \overline{0}$ in $R/\text{rad}(I)$ with $\overline{a} \neq 0$. Then $ab \in \text{rad}(I)$, but $a \notin \text{rad}(I)$. Since $R/\text{rad}(I)$ is domainlike, $\text{rad}(I)$ is primary. Therefore, $b^n \in \text{rad}(I)$ for some $n > 0$, which implies $(b^n)^l \in I$ for some $l > 0$. Thus, $b \in \text{rad}(I)$, $\overline{b} = \overline{0}$, and $R/\text{rad}(I)$ is an integral domain.

Given the above result, it is natural to consider when $R/\text{nil}(R)$ is présimplifiable.

Theorem 6. *Given a ring R, $R/\text{nil}(R)$ is présimplifiable if and only if whenever $xy = x$ and $x \notin \text{nil}(R)$, then $y \in U(R)$.*

Proof. ($\Leftarrow$) Suppose $\overline{xy} = \overline{x}$ and $\overline{x} \neq \overline{0}$. Thus, $x \notin \text{nil}(R)$ and $x - xy \in \text{nil}(R)$. This implies $[x(1-y)]^n = x^n(1-y)^n = 0$. Therefore $x^n[1 - y \cdot \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}y^{i-1}] = 0$, or $x^n = x^n y \cdot \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}y^{i-1}$. Since $x \notin \text{nil}(R)$, we have $x^n \neq 0$ and so $y \cdot \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}y^{i-1} \in U(R)$. Hence, $y \in U(R)$ and $\overline{y} \in U(R/\text{nil}(R))$.

($\Rightarrow$) Suppose $xy = x$ and $x \notin \text{nil}(R)$. Then $\overline{xy} = \overline{x}$ and $\overline{x} \neq \overline{0}$. This implies $\overline{y} \in U(R/\text{nil}(R))$, and after a calculation similar to one above, we obtain $y \in U(R)$.

Thus, if R is présimplifiable, then $R/\text{nil}(R)$ is présimplifiable and hence strongly associate. However, as Example 11 will show, $R/\text{nil}(R)$ being présimplifiable does not imply that R is présimplifiable or that R is strongly associate. In addition, it is interesting to note that if R is strongly associate, then $R/\text{nil}(R)$ need not be présimplifiable. For example, $R = \mathbb{Z}_3 \times \mathbb{Z}_3$ is strongly associate and $(1,0)\,(1,0) = (1,0)$, where $(1,0)^n \neq (0,0)$ and $(1,0) \notin U(R)$.

We have previously seen that domainlike rings are always présimplifiable and that the converse is not true in general. In fact, if we have a présimplifiable ring that is not domainlike, then we can say that it has a certain degree of complexity within its prime spectrum.

Lemma 7. *If R is présimplifiable and (0) is not primary, then $dim(R) \geq 1$.*

Proof. Since (0) is not primary, there exists $x, y \in R$ such that $xy = 0$ where $y \neq 0$ and $x^n \neq 0$ for all n. So, $x \in Z(R) \subseteq J(R)$ by Theorem 1, and hence x is contained in every maximal ideal of R. Now, $S = \{x^n\}_{n=1}^{\infty}$ is a multiplicatively closed set and $x \notin \text{nil}(R)$. Thus, $\text{nil}(R)$ is an ideal disjoint from S. Using Zorn's Lemma, expand $\text{nil}(R)$ to a prime ideal P disjoint from S. Then $x \notin P$, and so P is not a maximal ideal. Thus $P \subsetneq M$ for some maximal ideal M of R, which implies $dim(R) \geq 1$.

It follows directly from Lemma 7 that if R is présimplifiable and $dim(R) = 0$, then (0) must be primary and hence R is domainlike.

Theorem 8. *For a ring R, the following are equivalent.*

(1) *R is présimplifiable with $dim(R) = 0$;*

(2) *R has a unique prime ideal, i.e. R is primary;*

(3) *R is domainlike and all nonunits are zero-divisors.*

Proof. (1) $\Rightarrow$ (2) Since $dim(R) = 0$, all prime ideals are maximal. By Lemma 7, since R is présimplifiable and $dim(R) = 0$, we must have $\text{nil}(R)$ is prime and hence maximal. Since $\text{nil}(R)$ is maximal and is the intersection of all prime ideals of R, we have $\text{nil}(R)$ as the only prime ideal in R. Hence, R is primary.

(2) $\Rightarrow$ (3) By Lemma 4.

(3) $\Rightarrow$ (1) If R is domainlike, then R is présimplifiable. Also, if R is domainlike, then $Z(R)$ is the unique minimal prime ideal of R. Since all non-units are zero-divisors, we have that $Z(R)$ is the unique prime ideal.

Now, consider a domainlike ring R with $dim(R) = 0$. By Lemma 4, R has a unique minimal prime, namely $\text{nil}(R)$. The nilradical is contained in every prime ideal, and since $dim(R) = 0$, we see that R has a unique prime ideal and hence is quasi-local. However, R need not be Noetherian. Consider the classic example $R = K[x_1, x_2, \ldots]/(x_2^2, x_3^3, \ldots)$ for some field K. The only prime ideal is the image of $(x_1, x_2, x_3, \ldots)$, and hence $dim(R) = 0$. Since $Z(R)$ is always a union of prime ideals, we see that $Z(R) = (\overline{x_2}, \overline{x_3}, \ldots) = \text{nil}(R)$, and thus R is domainlike. However, this ideal is not finitely generated, so R is not Noetherian.

Note that if R has a unique minimal prime ideal (for example if R is domainlike) and R is Artinian, then we have $dim(R) = 0$, R is Noetherian, R is local, and $\text{nil}(R)$ is nilpotent.

It has been shown in [2] and [30] that if R is domainlike, then every subring of R is also domainlike (i.e. domainlike implies hereditarily domainlike), and $R[X]$ is domainlike. If R is domainlike and Noetherian, then $R[[X]]$ is also domainlike, as the following proposition shows.

Proposition 9. *Let R be a Noetherian ring. Then R is domainlike if and only if $R[[X]]$ is domainlike.*

Proof. ($\Leftarrow$) Suppose $ab = 0$ in R with $b \neq 0$. Then $ab = 0$ in $R[[X]]$, and $a^n = 0$ for some n.

($\Rightarrow$) Let $f = \sum a_i x_i \in Z(R[[X]])^*$. Since R is Noetherian, there exists an $r \in R^*$ such that $rf = 0$. Hence, $ra_i = 0$ for all i. Since R is domainlike and $a_i \in Z(R)$, we have $a_i \in \text{nil}(R)$ for all i. Thus, $f \in \text{nil}(R[[X]])$, since R is Noetherian.

Note that the Noetherian condition is not necessary to prove that R is domainlike whenever $R[[X]]$ is domainlike. However, the following example shows that the converse is false in general if R is not Noetherian.

Example 10. Let $R = \mathbb{Z}[x_1, x_2, \ldots]/(x_1^2, x_2^3, x_3^4, \ldots, x_1x_2, \ldots, x_1x_k, \ldots)$. Let $f \in R$. If f has no constant term, it will be nilpotent. If f has a constant term, it cannot be a zero-divisor. Hence, $Z(R) \subseteq \text{nil}(R)$, and so R is domainlike. Let $g = \overline{x_2} + \overline{x_3}Y + \overline{x_4}Y^2 + \overline{x_5}Y^3 + \cdots \in R[[Y]]$. Then $\overline{x_1}g = 0$, but g is not nilpotent. Thus, $Z(R[[Y]]) \nsubseteq \text{nil}(R[[Y]])$, and $R[[Y]]$ is not domainlike.

A final observation is that the domainlike property is not preserved under direct or subdirect products, since présimplifiable rings have no nontrivial idempotents.

3. Idealizations and localizations

Given a unitary R-module M, the *idealization of M in R*, denoted by $R(+)M$, is the set $\{(r, m) \mid r \in R, m \in M\}$, with addition defined componentwise and $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. In $R(+)M$, it is straightforward to check that $U(R(+)M) = \{(r, m) \mid r \in U(R)\}$, and by [10, Proposition 1.1] we have $Z(R(+)M)^* = \{(0, m) \mid m \in M^*\} \cup \{(a, n) \mid a \in R^*, n \in M \text{ and for some } m \in M^*, am = 0\} \cup \{(a, n) \mid a \in Z(R)^*, n \in M\}$. We start with an example referred to in the previous section.

Example 11. Consider the idealization $R = \mathbb{Z}(+)\mathbb{Z}_8$. By Theorems 14 and 15 of [2], R is not a strongly associate (and hence not a présimplifiable) ring. We have $\text{nil}(R) = \{(0, b) \mid b \in \mathbb{Z}_8\}$. Using Theorem 6, suppose that $(a, b)(y_1, y_2) = (a, b)$ and

$(a,b) \notin \text{nil}(R)$, i.e. $a \neq 0$. Then $(ay_1, ay_2 + by_1) = (a,b)$, which implies $ay_1 = a$. So, $y_1 = 1$, and $(y_1, y_2) \in U(R)$. Thus, $R/\text{nil}(R)$ is présimplifiable. So, $R/\text{nil}(R)$ being présimplifiable does not imply that R is présimplifiable.

Definition 12. An R-module M *preserves* $Z(R)$ if $rm = 0$ with $m \neq 0$ implies $r \in Z(R)$.

Given the definition above, we find a characterization of when an idealization is domainlike.

Theorem 13. *Let R be a ring and let M be an R-module. Then $R(+)M$ is domainlike if and only if R is domainlike and M preserves $Z(R)$.*

Proof. ($\Rightarrow$) Let $a, b \in R$ with $ab = 0$ and $a \neq 0$. Then $(a,0)(b,0) = (0,0)$ with $(a,0) \neq (0,0)$. Since $R(+)M$ is domainlike, we have $(b,0)^n = (0,0)$ for some $n > 0$. This implies that $b^n = 0$. Thus, (0) is primary in R and hence R is domainlike. Now, assume that for some $m \in M^*$ and some $r \in R^*$ we have $rm = 0$. Therefore, $(r,0)(0,m) = (0,0)$ and $(0,m) \neq (0,0)$. Since $R(+)M$ is domainlike, it follows that $(r,0)^n = (0,0)$ for some $n > 0$. So $r^n = 0$, and hence $r \in Z(R)$.

($\Leftarrow$) Let $(a,l)(b,m) = (0,0)$ with $(a,l) \neq (0,0)$. If $a \neq 0$, then $b^n = 0$ for some $n > 0$ because R is domainlike. Hence, $(b,m)^{2n} = (b,m)^n (b,m)^n = (0,k)(0,k) = (0,0)$. If $a = 0$, then $bl = 0$ and $l \neq 0$. So, $b \in Z(R)$. Hence, $b^n = 0$ for some $n > 0$. Again, $(b,m)^{2n} = (0,0)$. Thus, $R(+)M$ is domainlike.

Though Theorem 13 shows that if $R(+)M$ is domainlike then R is domainlike, the converse is not true in general.

Example 14. Let $R = \mathbb{Z}$ and consider $\mathbb{Z}(+)\mathbb{Z}_2$. Clearly $\mathbb{Z}$ is domainlike. However, $\mathbb{Z}(+)\mathbb{Z}_2$ is not domainlike, since $(0,1)(2,1) = (0,0)$ and $(0,1) \neq (0,0)$, yet $(2,1)$ is not nilpotent.

The next theorem is a classical result stated in terms of the domainlike property.

Theorem 15. *Any localization of a domainlike ring is domainlike.*

The converse is clearly false, since the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not domainlike, yet every localization is a domain and hence domainlike. Theorem 15 can be strengthened with the introduction of a curious property.

Proposition 16. *Let R be a ring, and let $a, b \in R^*$. Assume that whenever $ab = 0$ there exists a proper ideal I containing $ann(a)$ and $ann(b^i)$ for all $i > 0$. Then R is domainlike if and only if R_P is domainlike for every prime ideal P.*

Proof. ($\Rightarrow$) Theorem 15.

($\Leftarrow$) Assume R_P is domainlike for every prime ideal P. Suppose $ab = 0$ with $a \neq 0$. Let P be a prime ideal containing $ann(a)$ and $ann(b^i)$ for all $i > 0$. In R_P we have

$\frac{a}{1} \cdot \frac{b}{1} = \frac{0}{1}$ with $\frac{a}{1} \neq \frac{0}{1}$, since there is no $s \in R \backslash P$ such that $sa = 0$. Since R_P is domainlike, we have $(\frac{b}{1})^n = \frac{b^n}{1} = \frac{0}{1}$ for some n. Then $b^n = 0$, since there is no $t \in R \backslash P$ such that $tb^n = 0$. Thus, R is domainlike.

Though Proposition 16 features an odd condition, it is worth noting that any ring in which $Z(R)$ is contained in a proper ideal (for example, if R is quasilocal) will satisfy this condition.

4. Zero-divisor graphs of domainlike rings

The concept of the graph of the zero-divisors of a ring was first introduced by Beck in [12] when discussing the coloring of a commutative ring. In his work, all elements of the ring were vertices of the graph. D.D. Anderson and Naseer used this same concept in [3]. We adopt the approach used by D.F. Anderson and Livingston in [8] and consider only the nonzero zero-divisors as vertices of the graph.

For the sake of completeness, we state some definitions and notations. The zero-divisor graph of R, denoted $\Gamma(R)$, is the graph whose set of vertices is $Z(R)^*$, and for distinct $r, s \in Z(R)^*$, there is an edge connecting r and s if and only if $rs = 0$. We represent this edge by $r - s$. For two distinct vertices a and b in a graph Γ, the distance between a and b, denoted $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, $d(a, b) = \infty$. The *diameter* of a graph Γ is $\operatorname{diam}(\Gamma) = \sup \{d(a, b) \mid a \text{ and } b \text{ are distinct vertices of } \Gamma\}$. The *girth* of a graph Γ, denoted $\operatorname{g}(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle; otherwise, $\operatorname{g}(\Gamma) = \infty$. A graph is said to be *connected* if there exists a path between any two distinct vertices, and a graph is *complete* if it is connected with diameter one. A singleton graph is connected and has diameter zero.

D.F. Anderson and Livingston in [8], Mulay in [27], and DeMeyer and Schneider in [20] examined, among other things, the diameter and girth of the zero-divisor graph of a commutative ring. For instance, Anderson and Livingston showed the zero-divisor graph of a commutative ring is connected with diameter less than or equal to three [8, Theorem 2.3]. In addition, they showed that the girth is either infinite or less than or equal to four when R is Artinian and conjectured this would hold in general. DeMeyer and Schneider, and Mulay proved this conjecture independently, and a short proof can be found in [9].

The area of zero-divisor graphs has received a great deal of attention during the past few years. Many of the papers focus on the behavior of zero-divisor graphs of specific algebraic structures, or on the zero-divisor graphs of rings with particular properties. For example, the zero-divisor graphs of the rings of polynomials and power series over commutative rings were examined in [9], while the zero-divisor graphs of idealizations and direct products of rings were discussed in [10] and [11], respectively. In [7] and [25], the graphs of Von Neumann regular rings were studied, while the behavior of the zero-divisor graph of abelian regular rings was covered in [26]. A directed zero-divisor graph for noncommutative rings has been studied extensively in [1], [28], and [29], and zero-divisor graphs have been applied to the more general settings of nearrings and semigroups in [19] and [18]. In this section we examine the structure of the zero-divisor graphs of domainlike rings.

An examination of the zero-divisor graph of some domainlike rings quickly reveals that both the girth and the diameter are even more restricted than the already restrictive bounds presented above. In this section, we will assume that all domainlike rings being considered are not integral domains.

Theorem 17. *For a domainlike ring R, diam$(\Gamma(R)) \leq 2$.*

Proof. Let $a, b \in Z(R)^*$ with $d(a, b) > 1$. Since a and b are nilpotent, let m and n be the least positive integers such that $a^m = 0 = b^n$. Let i and j be positive integers such that $a^i b^j \neq 0$, but $a^{i+1} b^j = 0$ and $a^i b^{j+1} = 0$. Clearly, $a - a^i b^j - b$, and hence $d(a, b) = 2$.

When the diameter of $\Gamma(R)$ is 0 or 1 for a domainlike ring R, we see that $Z(R)^2 = 0$ by [8, Theorem 2.8]. A domainlike ring with diam$(\Gamma(R)) = 2$ need not exhibit any such property. For example, diam$(\Gamma(K[x_1, x_2, \ldots]/(x_1^2, x_2^3, \ldots)))$ for any field K, while $Z(K[x_1, x_2, \ldots]/(x_1^2, x_2^3, \ldots))^n \neq 0$ for all n.

In terms of the girth of a zero-divisor graph, domainlike rings show almost complete uniformity in assuming the minimum possible girth of 3. In fact, there are only three domainlike rings whose zero-divisor graphs contain three or more vertices and do not contain a cycle. Additionly, there are only four domainlike rings whose zero-divisor graphs contain fewer than three vertices and hence contains no cycles.

Lemma 18. [31, Lemma 4.2] *Let $\Gamma(R)$ be a zero-divisor graph with vertices a and b such that $a - b$ and $a^2 = 0 = b^2$. If diam$(\Gamma(R)) > 1$, then $g(\Gamma(R)) = 3$.*

Proof. Assume that a, $b \in Z(R)^*$ with $a - b$, diam$(\Gamma(R)) > 1$ and $a^2 = b^2 = 0$. Assume that there is no 3-cycle in $\Gamma(R)$. Since diam$(\Gamma(R)) > 1$ there exists some $c \in Z(R)^* \setminus \{a, b\}$ such that (without loss of generality) $ac = 0 \neq bc$. Since $a\,(a + b) = 0 = b\,(a + b)$ and $c\,(a + b) \neq 0$, we see that $a + b \in Z(R)^*$. Therefore $a - b - (a + b) - a$ is a 3-cycle, a contradiction.

Lemma 19. *Let $\Gamma(R)$ be a zero-divisor graph with vertices a and b such that $a - b$, $a^3 = 0 = b^3$, and $a^2, b^2 \neq 0$. Then $g(\Gamma(R)) = 3$.*

Proof. Suppose $a - b$, $a^3 = 0 = b^3$, and $a^2, b^2 \neq 0$. Then $ab^2 = 0$, and $b^2 \neq a$, lest $b^4 = 0 = a^2$. Clearly, $b^2 \neq b$ or 0. Thus, we get the 3-cycle $b - a - b^2 - b$, and $\mathrm{g}(\Gamma(R)) = 3$.

Lemma 20. *If there exists an $a \in Z(R)^*$ with $a^n = 0$ and $a^{n-1} \neq 0$ for some $n \geq 4$, then $g(\Gamma(R)) = 3$.*

Proof. If there exists some $a \in Z(R)^*$ with $a^n = 0$ and $a^{n-1} \neq 0$ for some $n \geq 5$, then $a^{n-3} - a^{n-2} - a^{n-1} - a^{n-3}$, and hence $\mathrm{g}(\Gamma(R)) = 3$.

If there exists some $a \in Z(R)^*$ with $a^4 = 0$ and $a^3 \neq 0$, then consider the element a^2+a^3. We have that $a^2+a^3 \neq a^2, a^3$. If $a^2+a^3 = 0$, then $a^2 = -a^3$, which implies $a^3 = a \cdot a^2 = a(-a^3) = -a^4 = 0$, a contradiction. Thus, we get the cycle $a^2 - a^3 - (a^2+a^3) - a^2$. Thus, $\mathrm{g}(\Gamma(R)) = 3$.

Theorem 21. *Let R be a domainlike ring that is not an integral domain. Then either $g(\Gamma(R)) = 3$ or ∞. Moreover, $g(\Gamma(R)) = \infty$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_4$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, or $\mathbb{Z}_4[x]/(2x, x^2-2)$.*

Proof. By Theorem 17, $\mathrm{diam}(\Gamma(R)) \leq 2$ for domainlike rings. If $\mathrm{diam}(\Gamma(R)) = 0$, then by [6, Example 2.1] we have $R \cong \mathbb{Z}_2[x]/(x^2)$ or $\mathbb{Z}_4$, and $\mathrm{g}(\Gamma(R)) = \infty$. If $\mathrm{diam}(\Gamma(R)) = 1$, then $\mathrm{g}(\Gamma(R)) = 3$ if and only if $|Z(R)^*| \geq 3$. Again, by [6, Example 2.1] $\mathbb{Z}_9$ and $\mathbb{Z}_3[x]/(x^2)$ are the only domainlike rings whose zero-divisor graphs have a diameter of 1 on fewer than three vertices.

For the remainder of the proof we will assume that $\mathrm{diam}(\Gamma(R)) = 2$. If $|Z(R)^*| = 3$, then by [6, Example 2.1] we have $R \cong \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, or $\mathbb{Z}_8$, and $\mathrm{g}(\Gamma(R)) = \infty$.

So, assume $|Z(R)^*| \geq 4$ and let $a \in Z(R)^*$. Since $Z(R) \subseteq \mathrm{nil}(R)$, there exists an n such that $a^n = 0$, but $a^{n-1} \neq 0$. If $n \geq 4$, then by Lemma 20 $\mathrm{g}(\Gamma(R)) = 3$.

Now suppose that for all $a \in Z(R)^*$ we have $a^3 = 0$. Choose $a, b \in Z(R)^*$ with $d(a, b) = 2$. Then there exists a $c \in Z(R)^*$ such that $a - c - b$ is a path. If $a^2 = 0$ and $b^2 \neq 0$, then either $a - c$ and $c^2 = 0$, or $c - b$ and $c^2 \neq 0$, and hence by Lemmas 18 and 19 we get $\mathrm{g}(\Gamma(R)) = 3$. If $a^2 = b^2 = 0$ and $c^2 = 0$, then again Lemma 18 gives $\mathrm{g}(\Gamma(R)) = 3$. If $a^2 = b^2 = 0$ and $c^2 \neq 0$, it cannot be the case that $c^2 = a$ and $c^2 = b$, since a and b are distinct vertices of $\Gamma(R)$. So, we have either $c - c^2 - a - c$ or $c - c^2 - b - c$ is a cycle, and $\mathrm{g}(\Gamma(R)) = 3$.

Suppose $a^2, b^2 \neq 0$. If $c^2 \neq 0$, then Lemma 19 gives $\mathrm{g}(\Gamma(R)) = 3$. So, assume $c^2 = 0$. If there exists an $x \in Z(R)^*$ such that $c \neq x$, $x^2 = 0$, and $x - a - c$ or $x - b - c$, then by an identical argument as the above paragraph, we have $\mathrm{g}(\Gamma(R)) = 3$. Since $Z(R)$ is an ideal, we have that $c + c \in Z(R)$. We have $(c+c)^2 = 0$ and $c + c \neq c$; if $c + c \neq 0$, let $x = c + c$, and we get $\mathrm{g}(\Gamma(R)) = 3$.

Now suppose $c + c = 0$. If either a^2 or b^2 is not equal to c, let $x = a^2$ (or b^2), and again we get $\mathrm{g}(\Gamma(R)) = 3$. Then, suppose $a^2 = c = b^2$. By assumption, $|Z(R)^*| \geq 4$, $\mathrm{diam}(\Gamma(R)) = 2$, and $x^3 = 0$ for all $x \in Z(R)^*$. There exists a $d \in Z(R)^*$ with $d - a$, $d - b$, or $d - c$. If $d - a$, if $d - b$, if $d - c$ and $d^2 = 0$, or if $d - c$ and $d^2 \neq c$, then we can appeal to previous cases to obtain $\mathrm{g}(\Gamma(R)) = 3$. Now suppose $d - c$ and $d^2 = c$. By assumption, $ab \neq 0$, and since R is présimplifiable, $ab \neq a, b$. Further, $(ab)^2 = a^2 b^2 = c^2 = 0$. Thus, $ab = c$, for otherwise we would let $x = ab$ above and have $\mathrm{g}(\Gamma(R)) = 3$. Similarly, $ad = bd = c$. So, $a(b - d) = 0$. If $b - d \neq c$, we again have $\mathrm{g}(\Gamma(R)) = 3$. So, suppose $b = d + c$. Similarly, $b(d - a) = 0$. Again, if $d - a \neq c$, we will have $\mathrm{g}(\Gamma(R)) = 3$. Now, if $d = a + c$, we have $b = d + c = a + c + c = a + 0 = a$, which is a contradiction. Thus, every case leads to $\mathrm{g}(\Gamma(R)) = 3$.

An interesting way of viewing this result is that there are only three domainlike rings whose zero-divisor graphs have sufficient vertices to form a cycle but do not, namely $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, and $\mathbb{Z}_4[x]/(2x, x^2 - 2)$. The zero-divisor graphs of all other domainlike rings with sufficiently many zero-divisors contain 3-cycles.

5. Ultraproducts of domainlike rings

We begin this section by recalling some of the basic definitions involved in the construction of ultraproducts. Let I be a nonempty set and let $\mathcal{P}(I) = \{A \mid A \subseteq I\}$. We say D is a **filter on** I if $D \subseteq \mathcal{P}(I)$ and

(1) $\emptyset \notin D$ and $D \neq \emptyset$,

(2) $A, B \in D$ implies $A \cap B \in D$, and

(3) $A \in D$ and $A \subseteq B$ implies $B \in D$.

A filter D on I is an *ultrafilter* if and only if for every $A \subseteq I$ either $A \in D$ or $I \backslash A \in D$, and not both by (1) and (2) above. Now, let $\{R_\alpha\}_{\alpha \in I}$ be a collection of commutative rings. Let F be an ultrafilter on I. The *ultraproduct of the* R_α*'s modulo* F, $\prod_{\alpha \in I} R_\alpha / F$, is defined as $\prod_{\alpha \in I} R_\alpha / \sim$ where $(a_i) \sim (b_i)$ if $\{i \in I \mid a_i = b_i\} \in F$.

Theorem 22. *Let I be an indexing set. For each $i \in I$, let R_i be a ring from the set $\{R_1, R_2, \ldots, R_m\}$. Let F be any ultrafilter on I, and assume that either $|I| < \infty$ or $|R_i| < \infty$ for each i. If R_i is domainlike for every $i \in I$, then $\prod_{\alpha \in I} R_i / F$ is domainlike.*

Proof. Suppose $(a_i)(b_i) = (0)$ in $\prod_{\alpha \in I} R_i / F$. So, $\{i \mid a_i b_i = 0\} \in F$. If $(a_i) \neq (0)$, then $\{i \mid a_i = 0\} \notin F$. Thus $\{i \mid a_i \neq 0\} \in F$, since F is an ultrafilter. Now, $a_i b_i = 0$ and $a_i \neq 0$ implies there exists $n_i \in \mathbb{N}$ such that $b_i^{n_i} = 0$ in R_i. Therefore $\{i \mid a_i \neq 0\} \subseteq \{i \mid b_i^{n_i} = 0 \text{ for some } n_i\} \in F$. Let $n = \max\{n_i\}_{i=1}^{m}$. Then $(b_i)^n = 0$, since $\{i \mid b_i^{n_i} = 0 \text{ for some } n_i\} \subseteq \{i \mid b_i^n = 0\} \in F$.

By considering $R_i = \mathbb{Z}[x_1, x_2, \ldots]/(x_1^2, x_2^3, \ldots, x_1x_2, x_1x_{3,\ldots})$ and $I = \{1, 2, 3, \ldots\}$, we see that the conditions on $|I|$ and $|R_i|$ for each i are necessary. Additionally, the converse to the above theorem is not necessarily true.

Example 23. Let $I = \{1, 2\}$ with ultrafilter $F = \{\{1\}, \{1, 2\}\}$, let $R_1 = \mathbb{Z}_4$, and let $R_2 = \mathbb{Z}_6$. Observe that $\prod_{\alpha \in I} R_i / F$ is domainlike, since the nonzero zero-divisors of $\prod_{\alpha \in I} R_i / F$ are of the form (a, b), where $a \in Z(R_1)^*$. Any such (a, b) is nilpotent because R_1 is domainlike, yet R_2 is not domainlike.

It is also interesting to note an arbitrary ultraproduct of domainlike rings need not be domainlike.

Example 24. Let $R_i = \mathbb{Z}[x] / (x^i)$ for $i \in I = \{2, 3, 4, 5, \ldots\}$. Since $Z(R_i) = \{xp(x) + (x^i)\} \subseteq nil(R_i)$, R_i is domainlike. Let $F = \{\{n, n+1, n+2, \ldots\} \mid n \geq 2\}$.

Now, $(x, x, x, \ldots)\left(x, x^2, x^3, \ldots\right) = (0)$ and $\left(x, x^2, x^3, \ldots\right) \neq (0)$. However, for all $n \in \mathbb{N}$, $(x, x, x, \ldots)^n \neq (0)$. Thus, $\prod_{\alpha \in I} R_i/F$ is not domainlike.

Recall that a filter D on I is called *principal* if for some $A \subseteq I$, $D = \{B \mid A \subseteq B \subseteq I\}$. We call A the generator of the filter. If $A = \{i\}$ for some $i \in I$, then we call i the *base element* of the principal filter. It is straightforward to verify that if the set I is finite, then any ultrafilter F on I is principal and has a base element.

Theorem 25. *Let I be an indexing set. Let F be any principal ultrafilter on I with a base element. Then $\prod_{\alpha \in I} R_i/F$ is hereditarily strongly associate if and only if R_j is hereditarily strongly associate, where j is the base element of the ultrafilter F.*

Proof. Without loss of generality, let $j = 1$ be the base element of our ultrafilter F.

($\Rightarrow$) Let S_1 be a subring of R_1. Consider the subring of $\prod_{\alpha \in I} R_i/F$ given by $A = \left\{(a) \in \prod_{\alpha \in I} R_i/F \mid \text{the } R_1\text{-component is from } S_1\right\}$. Since this is a subring of the direct product $\prod_{\alpha \in I} R_i$, its image is also a subring of the ultraproduct $\prod_{\alpha \in I} R_i/F$. Thus, A, is associate. Let $a \sim b$ in S_1. Now, $(a, 1, 1, \ldots) \sim (b, 1, 1, \ldots)$ in A, and since A is associate, there exists $(u) \in U(A)$ such that $(a, 1, 1, \ldots)(u) = (b, 1, 1, \ldots)$. Then there exists $(v) \in U(A)$ such that $(u)(v) = (1) = (1, 1, \ldots)$ in A. Therefore, $C = \{i \mid u_i v_i = 1_{R_i}\} \in F$. If $1 \notin C$, then $\{1\} \cap C = \emptyset \in F$, a contradiction. Hence, any unit of A contains a unit of S_1 in its first component. So, $(a, 1, 1, \ldots) \sim (b, 1, 1, \ldots)$ in A, and A associate implies there exists $u_1 \in U(S_1) \subseteq U(R_1)$ such that $au_1 = b$. Thus, R_1 is hereditarily strongly associate.

($\Leftarrow$) Assume R_1 is hereditarily strongly associate. Let S be a subring of $\prod_{\alpha \in I} R_i/F$, and let (a) and (b) be associate elements of S. Let $S_1 = \{a_1 \in R_1 \mid a_1 \text{ is the first component of some element of } S\}$. Since 1 is the base element of our ultrafilter, we see that S_1 is a subring of R_1 and is hence associate. Again, since 1 is the base element of our ultrafilter, $(a) \sim (b)$ implies $a_1 \sim b_1$ in S_1. Since S_1 associate, there exists some $u_1 \in U(S_1) \subseteq U(R_1)$ such that $a_1 u_1 = b_1$. Observe that in our ultraproduct $(u_1, 0, 0, \ldots) \in U(S)$ and $(a)(u_1, 0, 0, \ldots) = (b)$.

The proof of Theorem 25 can be easily generalized to show the following result.

Corollary 26. *Let F be a principal ultrafilter whose generator A is a set of finite cardinality. Then $\prod_{\alpha \in I} R_i/F$ is hereditarily strongly associate if and only if R_j is hereditarily strongly associate for every $j \in A$.*

As Theorem 25 suggests, even if we consider more general ultrafilters than principal ultrafilters, we see that an ultraproduct being hereditarily strongly associate does not imply that each constituent ring need be hereditarily strongly associate. The following example illustrates this.

Example 27. Let $I = \mathbb{N}$ and let our ultrafilter F on I be the ultrafilter containing the filter $D = \{\{n, n+1, n+2, \ldots\} \mid n \in \mathbb{N}\}$. Let R_1 be any non-hereditarily strongly associate ring, and let R_i be any non-trivial field for $i \neq 1$. It can be observed that

$\prod_{\alpha \in I} R_i/F$ is hereditarily strongly associate, since every element of F is of infinite cardinality.

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